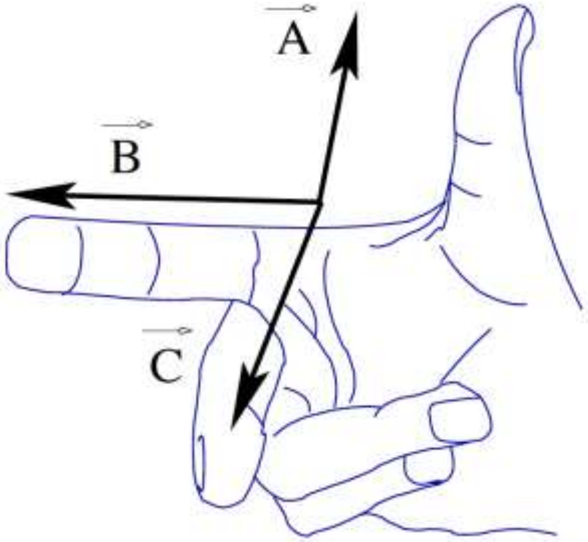
Physics 1

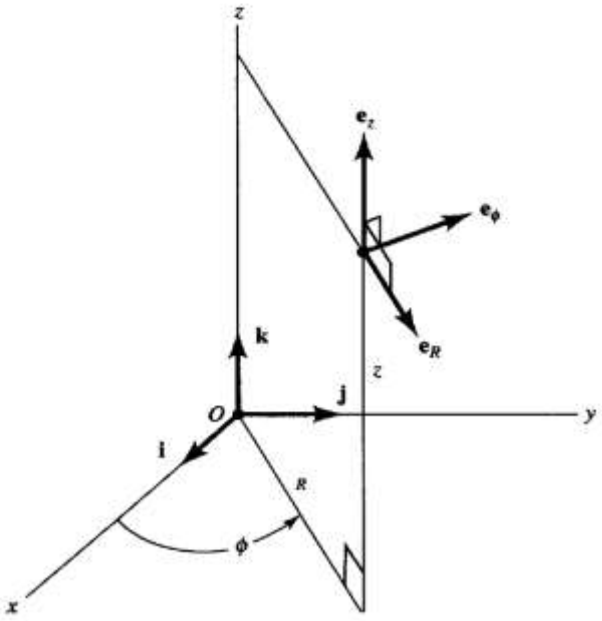
Chapter 1 : Mathematical Review

1. Dimensional Analysis

$$\boxed{[Q] = M^a \cdot L^b \cdot T^c \cdot I^d \cdot \theta^e \cdot N^f \cdot J^g}$$



2. Vectors



3. Coordinate Systems

I.1. What is Physics ?

- Physics is the study of all aspects of the universe.
- Physics is about understanding how everything works, from *nuclear reactors* to *nerve cells* to *spaceships*.
- Physics is an *experimental* science. Physists observe the phenomena of nature and try to find patterns and principles that relate these phenomina. These patterns are called physical theories or, when they are very well established and of broad use, physical laws or principles.
- The fundamental laws used in developing theories are expressed in the language of mathematics, the tool that provides a bridge between theory and experiment.

I.2. Physical quantity and Measurement

- A physical quantity is anything that can be measured and expressed as a number with a unit.
- For example, "the temperature is "$25°C$" or "the force is "$10\ N$“ both are physical quantities because they have a numerical value and a unit attached.
- To find precise relationships that describe physical phenomena we must be able to measure physical quantities such as lenght, area, volume, velocity, acceleration, mass, time and temperature…
- To do this we need units of measurment for all the quantities we are interested in.

- **Measurement** is the process of comparing a physical quantity with a chosen standard (the unit) to obtain a numerical value. It is the foundation of all experimental science.

- The result of measurement is always written as:

$$Quantity = Numerical\ value \times Unit$$

For example: $L = 12.5\ cm$, or $T = 300\ K$

I.3. Units and International System of units

- Units are the standardized references used to express the magnitude of a physical quantity. Without a unit, a number alone means nothing in physics
- Saying "$the\ length\ is\ 5$" is meaningless; "$the\ length\ is\ 5\ metres$" is precise and universally understood.
- We use the International System of units (also called the metric system or the SI system, for the French term Système International).
- The SI system is built on 7 base units, from which all other units are derived.

1.3.1. 7 base units of the SI system

Physical Quantity	SI Unit	Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

Example 1

In the following equations, the distance x is in meters, the time t is in seconds, and the velocity v is in meters per second. What are the **SI** units of the constants C_1 and C_2 ?

a) $x = C_1 + C_2 t$

b) $x = \frac{1}{2} C_1^2$

c) $V^2 = 2\, C_1\, x$

d) $x = C_1 \cos C_2 t$

e) $V = C_1 e^{-C_2 t}$

(*Hint:* The arguments of trigonometric functions and exponentials must be dimensionless. The "argument" of $\cos\theta$ is θ and that of e^x is x)

1.3.2. SI Derived Units

- All other physical quantities are expressed as combinations of these 7 base units. Some derived units are so commonly used they have their own names:

Quantity	Unit	Symbol	In base units
Force	newton	N	$kg \cdot m \cdot s^{-2}$
Energy	joule	J	$kg \cdot m^2 \cdot s^{-2}$
Power	watt	W	$kg \cdot m^2 \cdot s^{-3}$
Pressure	pascal	Pa	$kg \cdot m^{-1} \cdot s^{-2}$
Electric charge	coulomb	C	$A \cdot s$
Frequency	hertz	Hz	s^{-1}

Example 2

Newton's law of universal gravitation is represented by :

$$F = G\frac{M\,m}{r^2}$$

Here F is the magnitude of the gravitational force exerted by one small object on another,

M and m are the masses of the objects, and r is a distance.

Force has the **SI** units $kg \cdot m/s^2$.

What are the **SI** units of the proportionality constant G?

1.3.3. SI Prefixes

- SI uses a system of prefixes to express very large or very small quantities without writing many zeros:

Multiplication Factor	Prefix	Symbol
$1000\,000\,000\,000 = 10^{12}$	tera	T
$1000\,000\,000 = 10^{9}$	giga	G
$1000\,000 = 10^{6}$	mega	M
$1000 = 10^{3}$	kilo	k
$100 = 10^{2}$	hecto	h
$10 = 10^{1}$	deka	da
$0{,}1 = 10^{-1}$	deci	d
$0{,}01 = 10^{-2}$	centi	c
$0{,}001 = 10^{-3}$	milli	m
$0{,}000\,001 = 10^{6}$	micro	μ
$0{,}000\,000\,001 = 10^{-9}$	nano	n
$0{,}000\,000\,000\,001 = 10^{-12}$	pico	p
$0{,}000\,000\,000\,000\,001 = 10^{-15}$	femto	f
$0{,}000\,000\,000\,000\,000\,001 = 10^{-18}$	atto	a

1.4. Dimensional Analysis

- Dimensional analysis is a mathematical technique used to **check equations** and **solve problems** by tracking the dimensions (units) of physical quantities.
- The word dimension has a special meaning in physics. It denotes the physical nature of a quantity.
- Whether a distance is measured in units of feet or inches or meters, it is still a distance. We say its dimension is length.

1.4.1. Symbols in Dimensional Analysis

These are the fundamental symbols used to express dimensions :

Symbol	Dimension	Typical Unit
$\boldsymbol{M}$	Mass	kilogram (kg)
$\boldsymbol{L}$	Length	meter (m)
$\boldsymbol{T}$	Time	second (s)
$\boldsymbol{I}$	Electric Current	ampere (A)
$\boldsymbol{\theta}$	Temperature	kelvin (K)
$\boldsymbol{N}$	Amount of Substance	mole (mol)
$\boldsymbol{J}$	Luminous Intensity	candela (cd)

Example 3

What are the dimensions of the constants in each part of example 1?

1.4.3. Notation

Dimensions are written inside **square brackets** [ ] to mean *"the dimension of"*:

Notation	Meaning
$[x]$	"the dimension of x"
$[v] = LT^{-1}$	velocity has dimensions of Length/Time
$[F] = MLT^{-2}$	force has dimensions of Mass × Length / Time^2
$[E] = ML^2T^{-2}$	energy has dimensions of Mass × Length^2 / Time^2

Example 4

Show that the expression $v = at$ is dimensionally correct, where v represents speed, a acceleration, and t an instant of time.

1.4.4. Common Derived Dimension Expressions

- **Fill out the table below :**

Quantity	Symbol Expression
Speed / Velocity	
Acceleration	
Force	
Energy / Work	
Power	
Pressure	
Frequency	
Charge	
Density	
Momentum	

Quantity	Symbol Expression	Reads as
Speed / Velocity	LT^{-1}	Length per Time
Acceleration	LT^{-2}	Length per Time squared
Force	MLT^{-2}	Mass × Length per Time squared
Energy / Work	ML^2T^{-2}	Mass × Length squared per Time squared
Power	ML^2T^{-3}	Mass × Length squared per Time cubed
Pressure	$ML^{-1}T^{-2}$	Mass per Length per Time squared
Frequency	T^{-1}	per Time
Charge	IT	Current × Time
Density	ML^{-3}	Mass per Length cubed
Momentum	MLT^{-1}	Mass × Length per Time

1.4.5. What are the purposes of dimensional analysis?

Dimensional analysis serves the following purposes:

- Verification of the homogeneity of physical formulas.
- Determination of the nature of a physical quantity.
- Exploration of the general form of physical laws.

Example 5

Determine if the dimensions in each of the following equations are consistent.

See table 1.1.4 for definitions of symbols.

$$a)\ \ v^2 = at$$

$$b)\ \ x = vt + \frac{at^2}{2}$$

$$c)\ \ max + \frac{mv^2}{2} = Fx^2$$

$$d)\ \ v^2 = 2ax + \frac{x}{t}$$

$$e)\ \ v^2 = \frac{F\,x}{m}$$

Example 6

Determine whether or not each of the following equations is dimensionally correct.

M, L, and T indicate dimensions of mass, length, and time, respectively.

The symbols used in the equations and their dimensions are as follows:

Quantity	Dimensions	Equations
$F = force$	MLT^{-2}	$(a)\ \ v = a\,t$
$x = distance$	L	$(b)\ \ t = \sqrt{\dfrac{2\,x}{a}}$
$v\ = velocity$	LT^{-1}	$(c)\ \ v^3\ = 2\,a\,x^2$
$a = acceleration$	LT^{-2}	$(d)\ \ F = mvx$

Example 7

The pressure in a fluid in motion depends on its density ρ and its speed v.

Find a simple combination of density and speed that gives the correct dimensions of pressure.

1.4.6. Dimensional Homogeneity Symbol

Every physical quantity $\boldsymbol{Q}$ is fully described by 7 exponents: $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g})$.

Most quantities in mechanics only use the first three $(\boldsymbol{M}, \boldsymbol{L}, \boldsymbol{T})$.

$$\boxed{[Q] = M^a \cdot L^b \cdot T^c \cdot I^d \cdot \theta^e \cdot N^f \cdot J^g}$$

Example 8

Suppose we are told that the acceleration a of a particle moving with uniform speed v in a circle of radius r is proportional to some power of r, say r^n, and some power of v, say v^m.

Determine the values of n and m and write the simplest form of an equation for the acceleration.

Example 9

The position of a particle moving under uniform acceleration is some function of time and the acceleration. Suppose we write this position $\boxed{s = k\, a^m t^n}$, where k is a dimensionless constant.

1) Show by dimensional analysis that this expression is satisfied if $m = 1$ and $n = 2$.

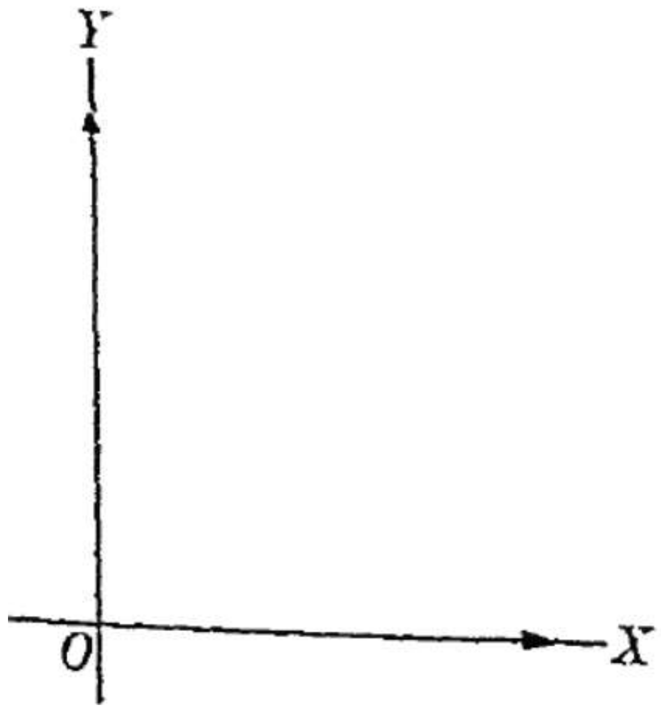
2) Can this analysis give the value of k?

2. Vectors

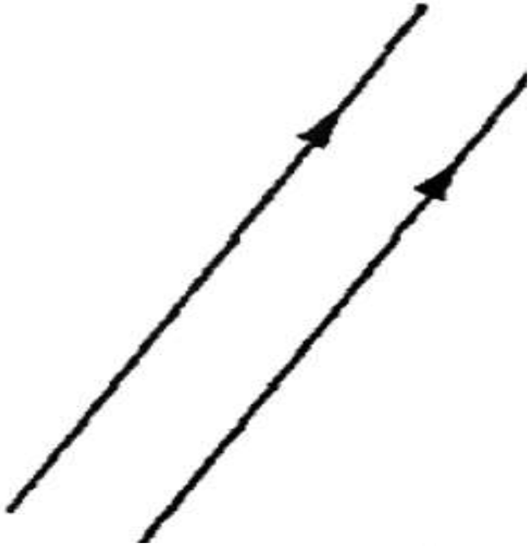
- The fundamental mathematical object used in Newtonian mechanics is the vector; therefore, the basic definitions and operations on vectors will be reviewed.
- Cartesian, cylindrical, and spherical coordinate systems are also presented.

2.1. Concept of Direction

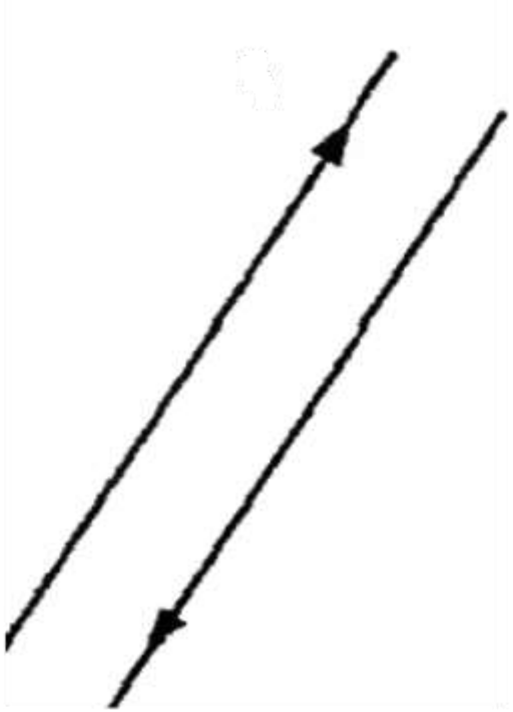
- When we are given a straight line, we can move along it in two opposite senses; these are distinguished by assigning to each a sign, plus or minus.
- Once the positive sense has been determined; we say that the line is oriented and call it an axis.
- The coordinate axes X and Y are oriented lines in which the positive senses are as indicated.
- The positive sense is usually indicated by an arrow. An oriented line or axis defines a direction.



- Parallel lines oriented in the same sense define the same direction



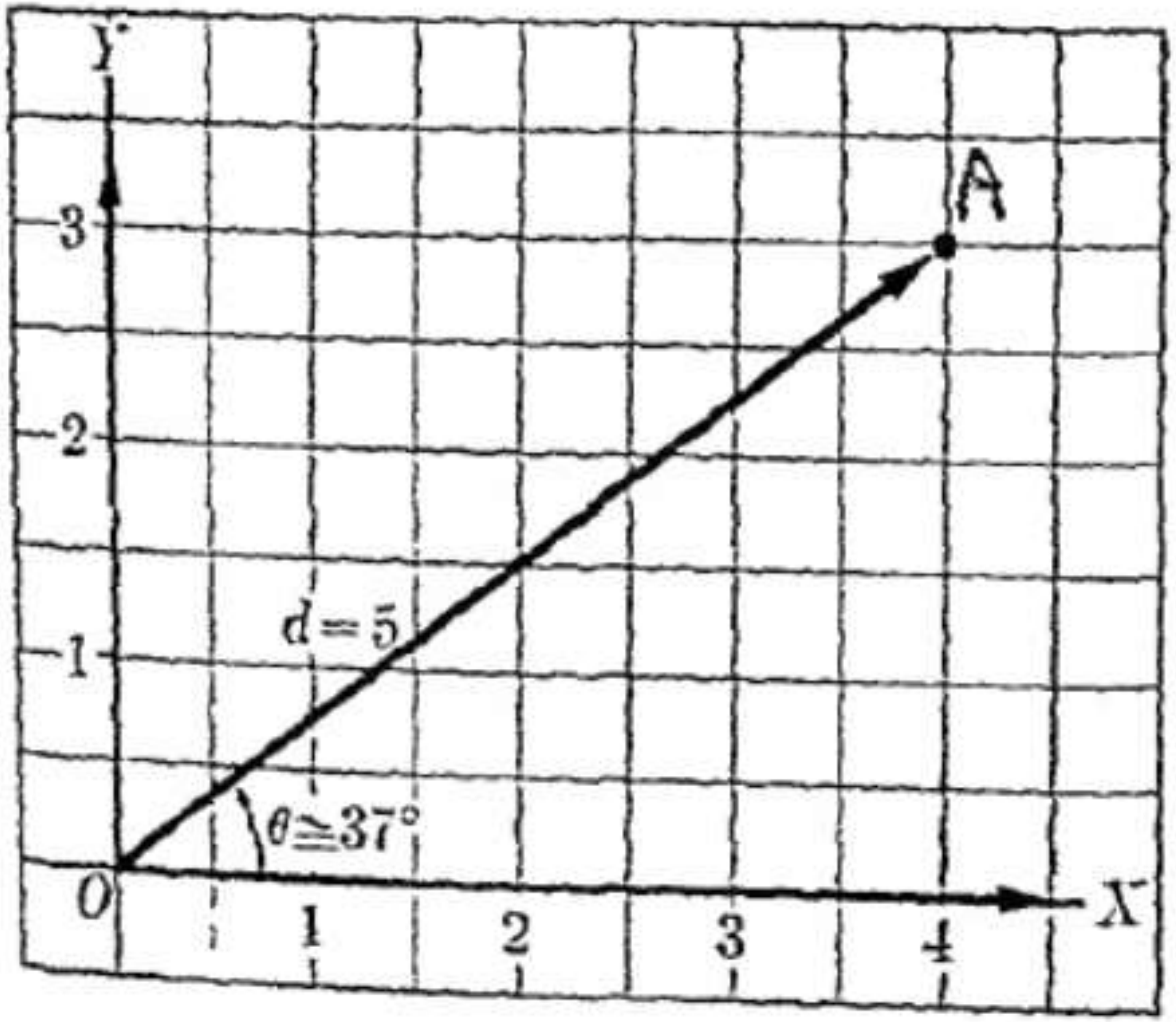
- but if they have opposite orientations they define opposite directions



2.2. Scalar and vector quantities

- A scalar quantity is described by a single number with a unit, for example, the mass of a person or the density of water.
- A vector quantity is one that has both a magnitude and a direction. Examples of vectors are displacement, velocity, acceleration, force and linear momentum...
- The magnitude of the displacement vector is called distance. The magnitude of the velocity vector is speed. For other vectors the magnitudes are not given special names. The magnitude of a vector is always a positive quantity.
- Operations with scalars follow the same rules as in elementary algebra. For example, $6kg + 3kg = 9kg$ or $4 \times 2s = 8s$.

- However, combining vectors requires a different set of operations. For example, if a particle is displaced from 0 to A, the displacement is determined by the distance $d = 5$ not 7.



- Graphically a vector is represented by an arrow OP defining the direction, the magnitude of the vector being indicated by the lenght of the arrow.

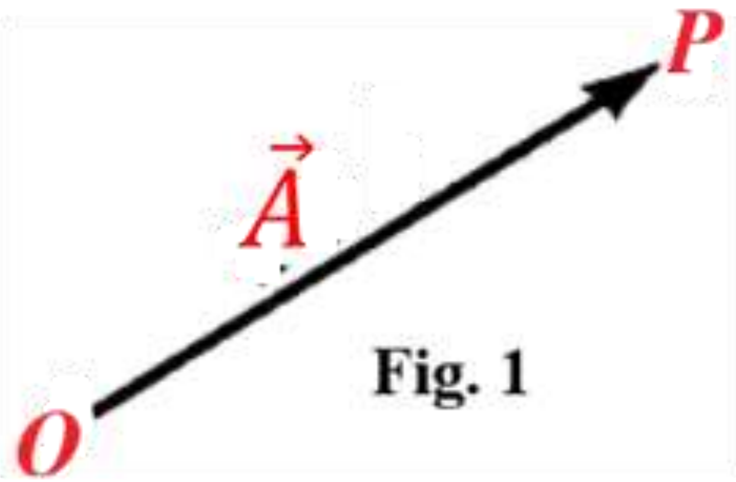


Fig. 1

- The taill end O of the arrow is called the origin or initial point of the vector, and the head P is called the terminal point or terminus.

- Analytically a vector is represented by a letter with an arrow over it as $\vec{A}$ in fig.1, and its magnitude is denoted by $|\vec{A}|$ or A. The vector $\vec{A}$ is also indicated as $\overrightarrow{OP}$ and its magnitude is denoted by $|\overrightarrow{OP}|$ or OP

- In three-dimensional space equipped with an orthonormal coordinate system

$\{OX, OY, OZ\}$, any vector $\vec{V}$ can be written as: $\vec{V} = x.\vec{\imath} + y.\vec{\jmath} + z.\vec{k}$ or $\vec{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- Here $\{x$,y and $z\}$ are the coordinates of the vector while $\left\{\vec{\imath}, \vec{\jmath}\ and\ \vec{k}\right\}$ are the basis vectors of the chosen coordinate system.

- The magnitude of a vector $\vec{V}$ is denoted $\left\|\vec{V}\right\|$ or V.

- In three-dimensional space, we have: $\left\|\vec{V}\right\| = \sqrt{x^2 + y^2 + z^2}$. "It is a quantity that is always positive."

Example 10

We consider the vectors

$$\overrightarrow{r_1} = 2\vec{i} + 3\vec{j} - \vec{k}$$

$$\overrightarrow{r_2} = 3\vec{i} - 2\vec{j} + 2\vec{k}$$

$$\overrightarrow{r_3} = 4\vec{i} - 3\vec{j} + 3\vec{k}$$

1. Calculate their magnitudes

2. Calculate the components and magnitudes of the vectors

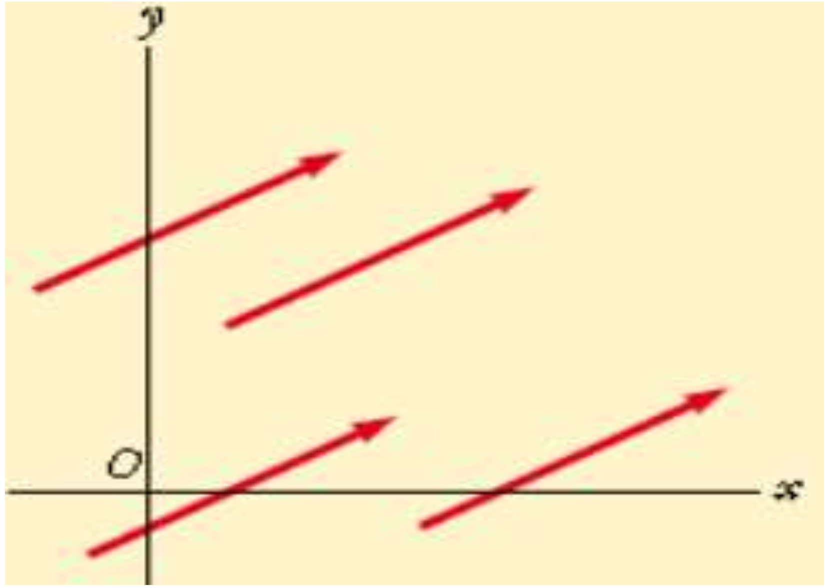
$$\vec{A} = \overrightarrow{r_1} + \overrightarrow{r_2} + \overrightarrow{r_3}$$

$$\vec{B} = \overrightarrow{r_1} + \overrightarrow{r_2} - \overrightarrow{r_3}$$

2.3. Some properties of vectors

➢Equality of Two Vectors :

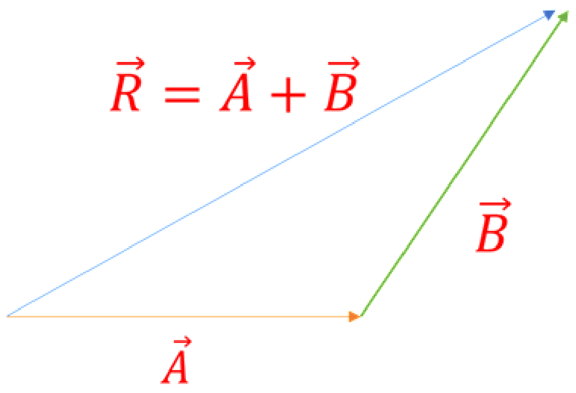
- For many purposes, two vectors $\vec{A}$ and $\vec{B}$ may be defined to be equal if they have the same magnitude and point in the same direction. That is, $\vec{A} = \vec{B}$ only if $A = B$ and if $\vec{A}$ and $\vec{B}$ point in the same direction along parallel lines.
- For example, all the vectors in the figure are equal even though they have different starting points.



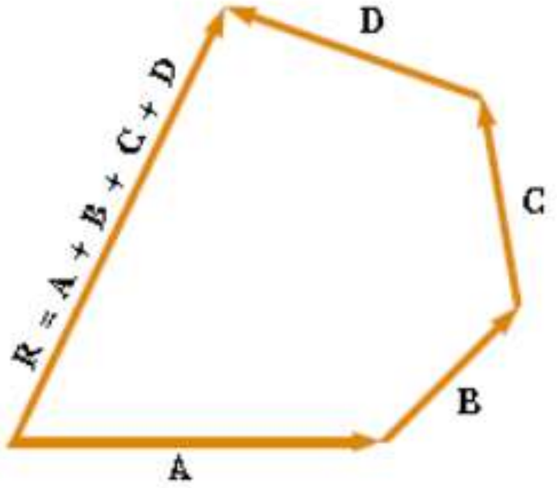
- This property allows us to move a vector to a position parallel to itself in a diagram without affecting the vector.

➢Adding Vectors

- The resultant vector $\vec{R} = \vec{A} + \vec{B}$ is the vector drawn from the tail of $\vec{A}$ to the tip of $\vec{B}$.



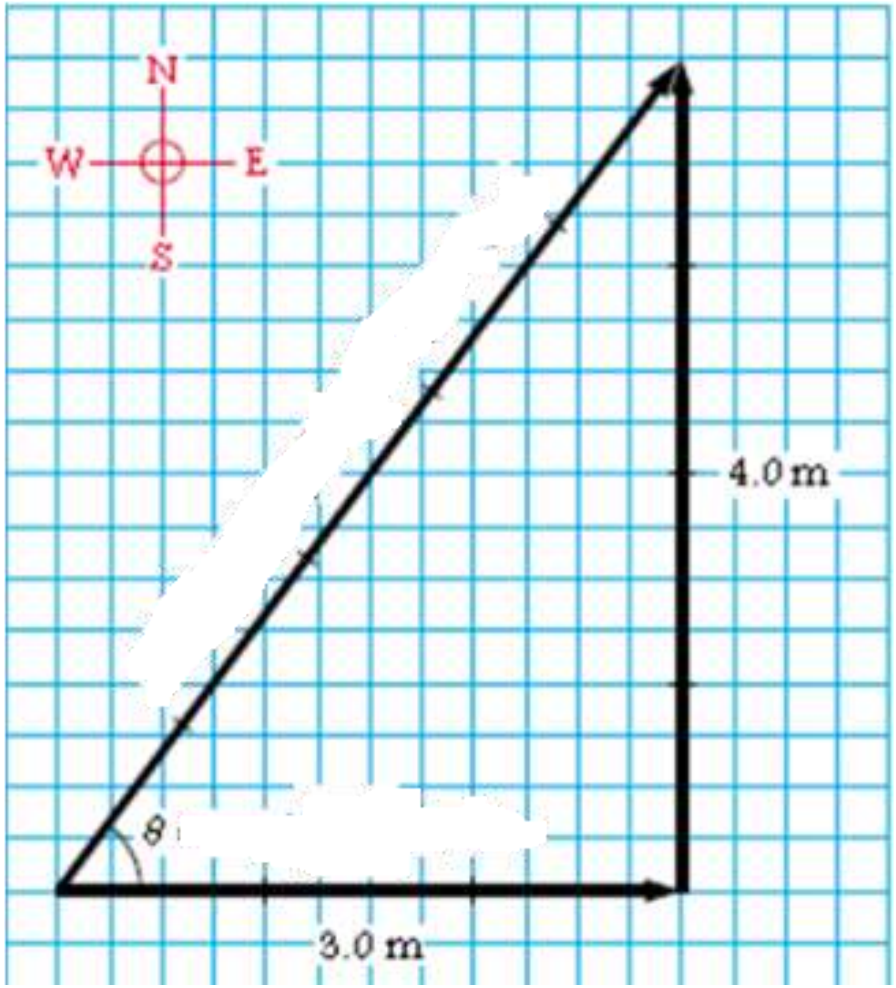
- The resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$ is the vector that completes the polygon.



- In other words, $\vec{R}$ is the vector drawn from the tail of the first vector to the tip of the last vector.

Example 11

If you walked 3.0 m toward the east and then 4.0 m toward the north, as shown in the Figure.



Find the magnitude and direction of your resultant displacement.

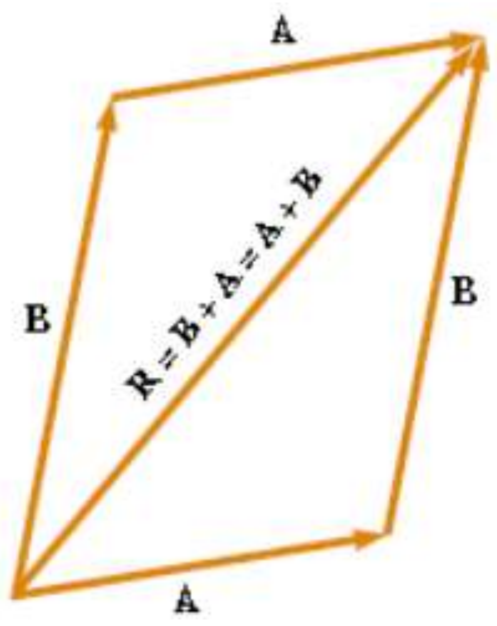
Example 12

A man walks eastward for 5 km and then northward for 10 km.

How far is he from his starting point?

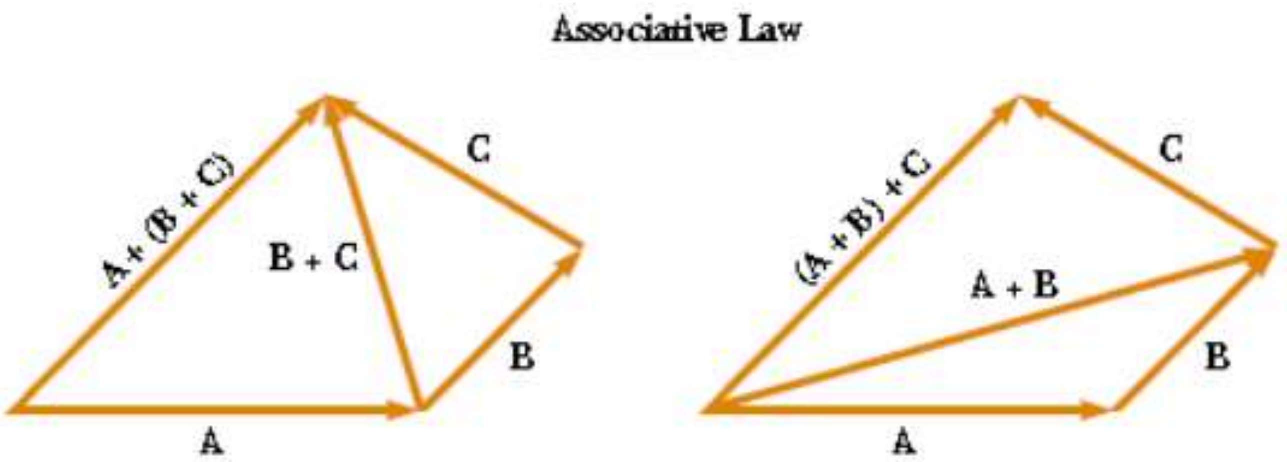
- **The commutative law of addition:**

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$



- **The associative law of addition:**

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

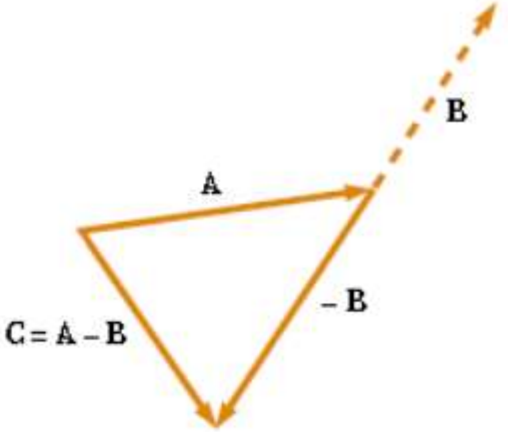


➢Negative of a Vector :

The negative of the vector $\vec{A}$ is defined as the vector that when added to $\vec{A}$ gives zero for the vector sum. That is, $\vec{A} + (-\vec{A}) = 0$. The vectors $\vec{A}$ and $-\vec{A}$ have the same magnitude but point in opposite directions.

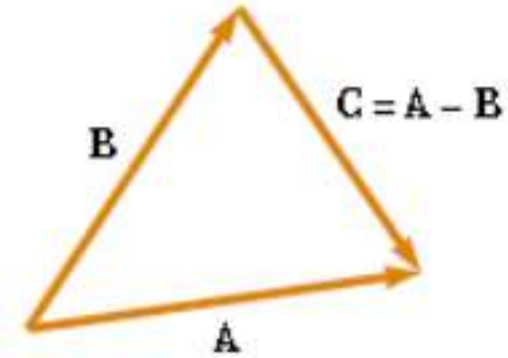
➢Subtracting Vectors :

We define the operation $\vec{A} - \vec{B}$ as vector $-\vec{B}$ added to vector $\vec{A}$:



$$- \vec{B} = \vec{A} + (-\vec{B})$$

➢In other words, the vector $\vec{A} - \vec{B}$ points from the tip of the second vector to the tip of the first.

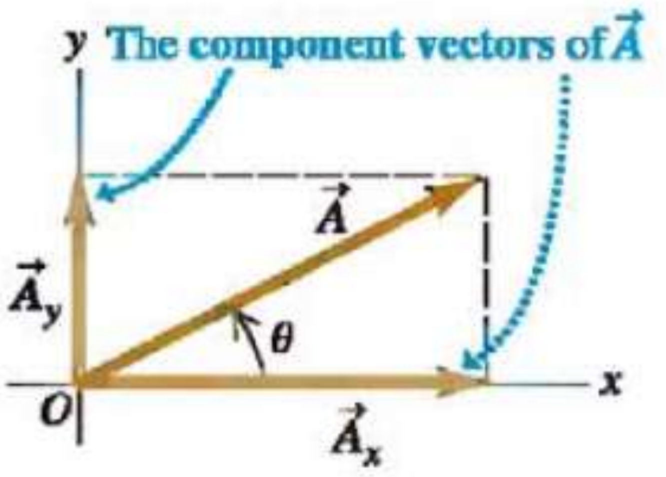


2.4. Components of a Vector

- In this section, we describe a method of adding vectors that makes use of the projections of vectors along coordinate axes.

- These projections are called the components of the vector. Any vector can be completely described by its components.

- Consider a vector $\vec{A}$ lying in the xy plane and making an arbitrary angle θ with the positive x axis, as shown in the figure.

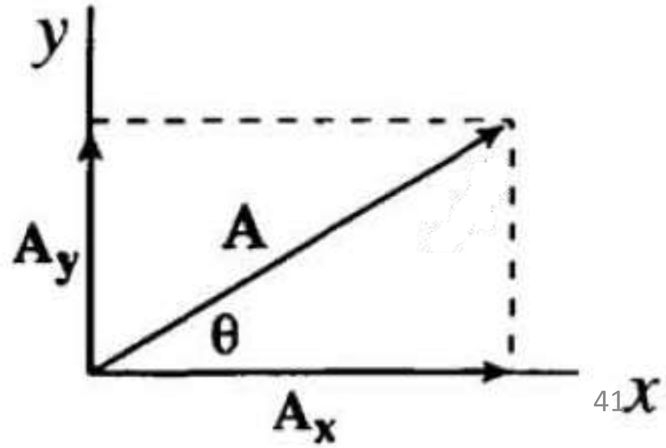


- The component A_x represents the projection of $\vec{A}$ along the x axis, and the component A_y represents the projection of $\vec{A}$ along the y axis.

- These components can be positive or negative. The component A_x is positive if A_x points in the positive x direction and is negative if A_x points in the negative x direction. The same is true for the component A_y .

- From the definition of sine and cosine, we see that $\cos\theta = \frac{A_x}{A}$ and that $\sin\theta = \frac{A_y}{A}$

  Hence, the components of $\vec{A}$ are $\begin{cases} A_x = A\cos\theta \\ A_y = A\sin\theta \end{cases}$



- These components form two sides of a right triangle with a hypotenuse of length A.

- Thus, it follows that the magnitude and direction of $\vec{A}$ are related to its components through the expressions

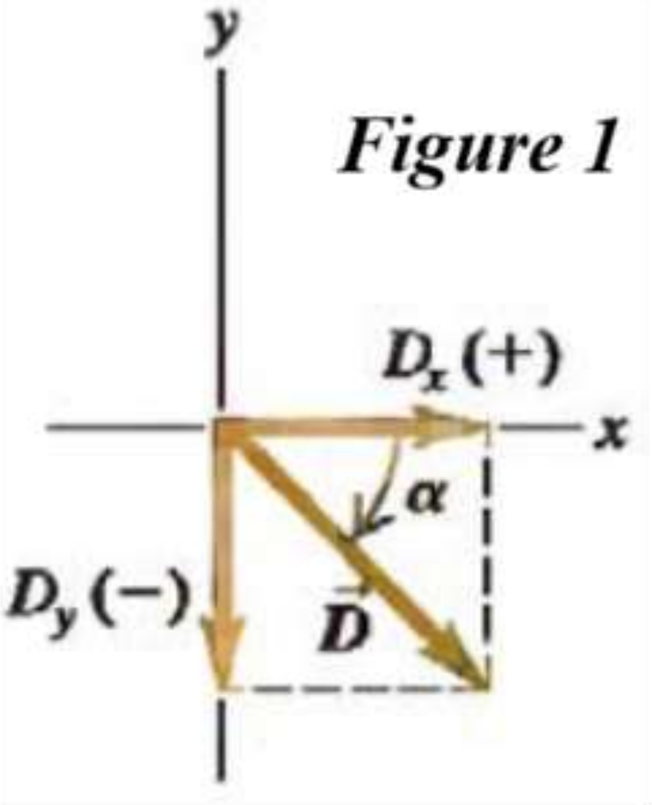
$$A = \sqrt{A_x^2 + A_y^2}$$

$$\tan\theta = \frac{A_y}{A_x}$$

Note that the signs of the components A_x and A_y depend on the angle θ.

Example 13

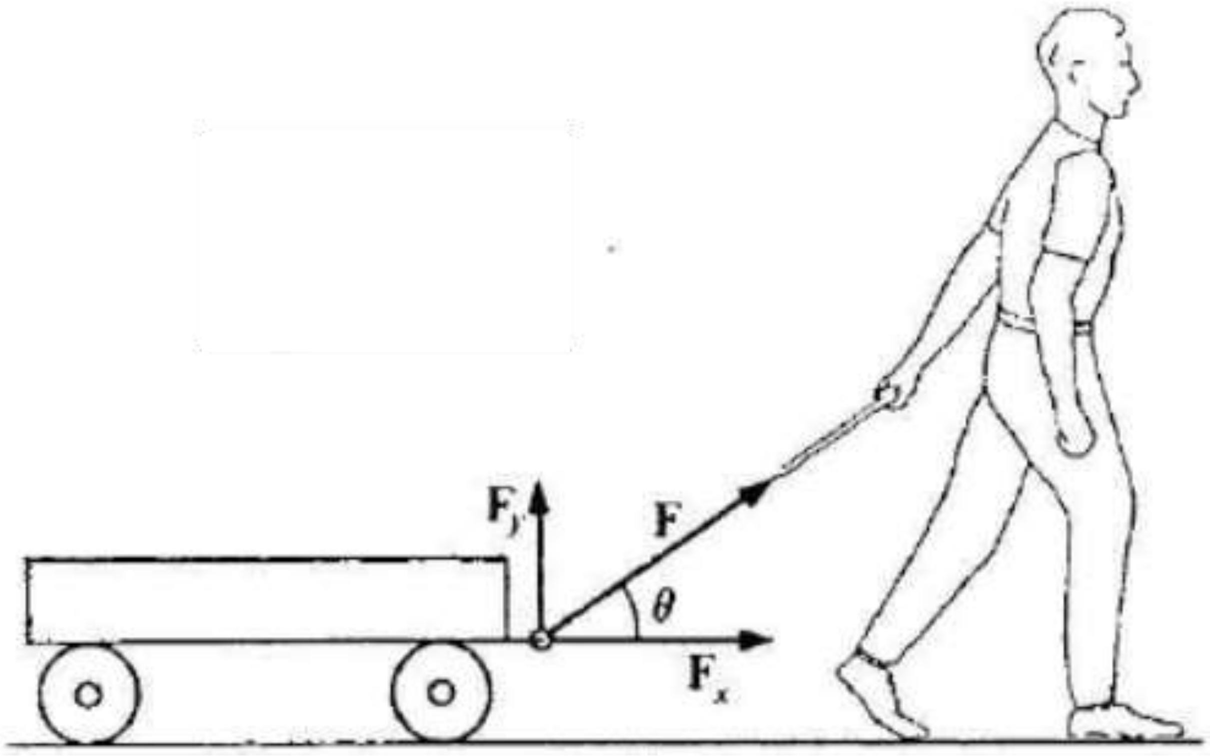
What are the x and y components (D_x, D_y) of vector $\vec{D}$ in figure 1?



The magnitude of the vector is $D = 3{,}00\ m$ and the angle $\alpha = 45°$.

Example 14

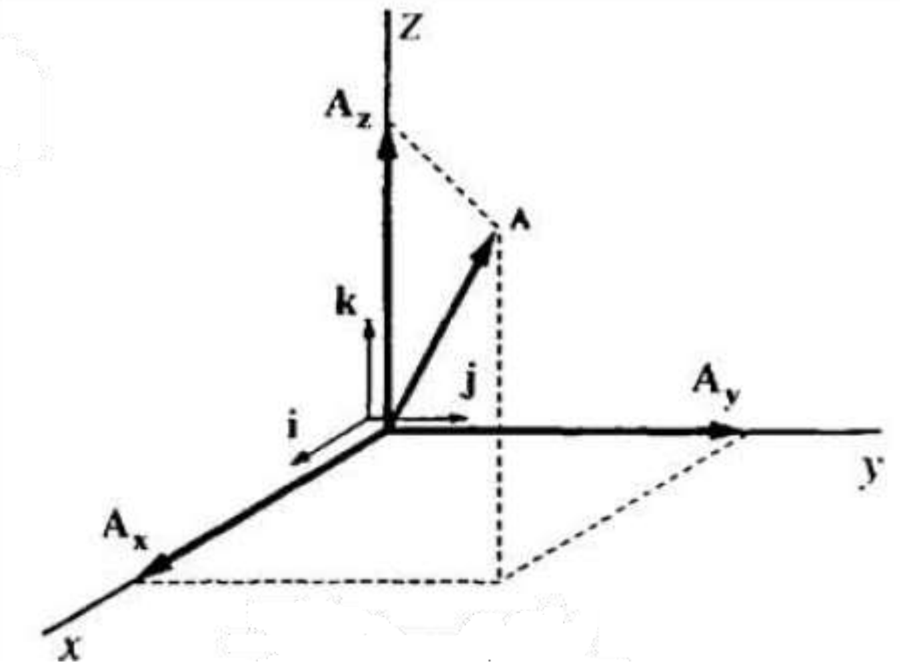
The man in the figure exerts a force of $100\ N$ on the wagon at an angle of $\theta = 30°$ above the horizontal.



Find the horizontal and vertical components of this force.

2.5. Unit Vectors

- Vector quantities often are expressed in terms of unit vectors.
- A unit vector is a dimensionless vector having a magnitude of exactly **1**.
- Unit vectors are used to specify a given direction and have no other physical significance. They are used solely as a convenience in describing a direction in space.
- When using Cartesian coordinates, we use the unit vectors $(\vec{i}, \vec{j}, \vec{k})$ pointing along the $(x\ , y, z)$ axes, respectively. Thus $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ as shown in Figure.



Example 15

A particle undergoes three consecutive displacements:

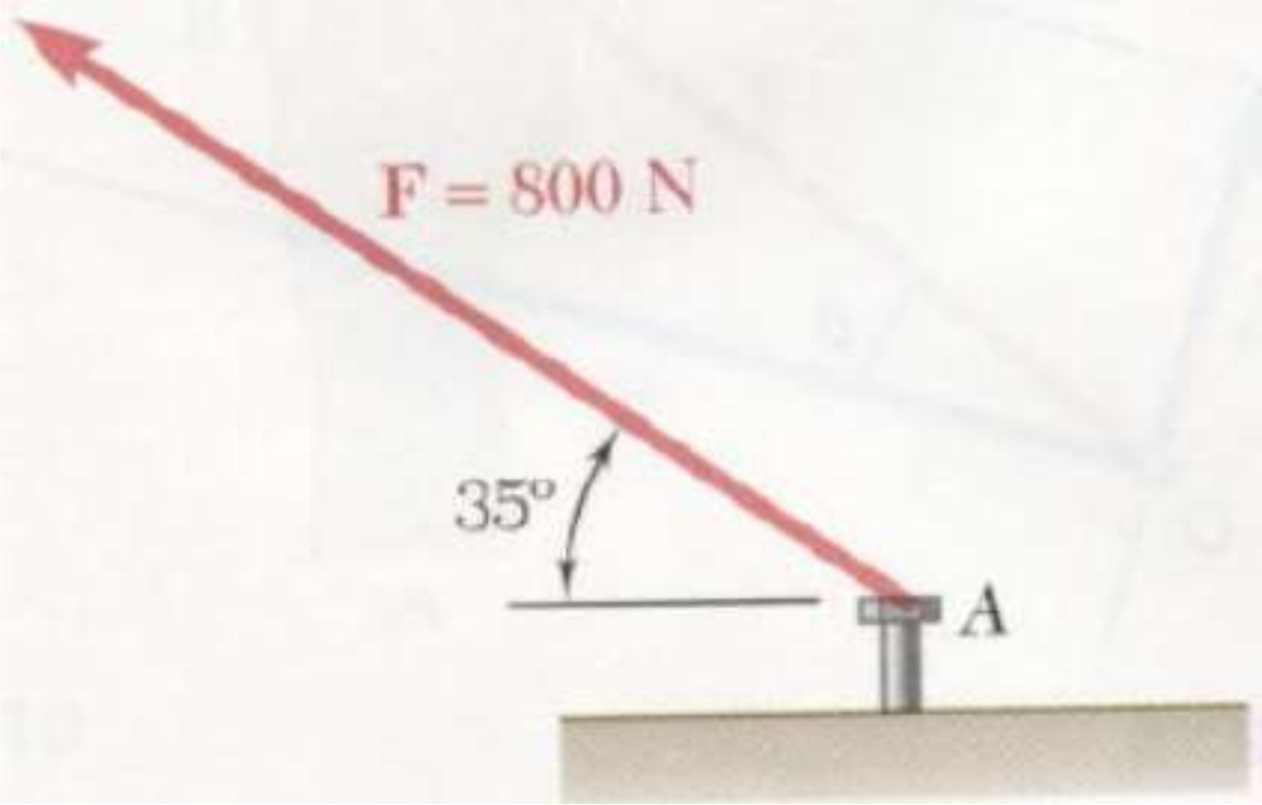
$\overrightarrow{d_1} = \left(15\,\vec{\imath} + 30\,\vec{\jmath} + 12\,\vec{k}\right) cm$; $\overrightarrow{d_2} = \left(23\,\vec{\imath} - 14\,\vec{\jmath} - 5\,\vec{k}\right) cm$

and $\overrightarrow{d_3} = (-13\,\vec{\imath} + 15\,\vec{\jmath})\, cm$

Find the components of the resultant displacement and its magnitude.

Example 16

A force of 800 *N* is exerted on a bolt *A* as shown in the figure :



Determine the horizontal and vertical component of the force.

Example 17

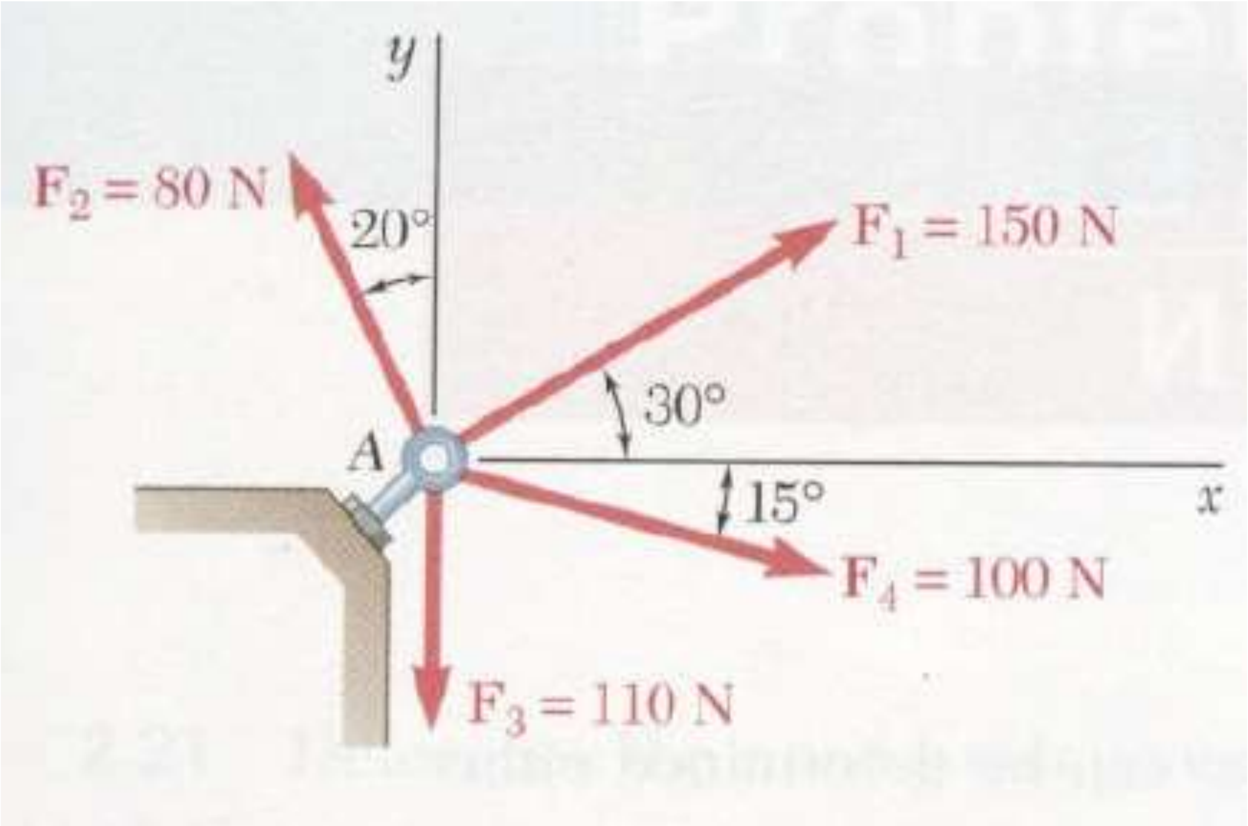
Given the two displacements

$\vec{D} = (6\,\vec{i} + 3\,\vec{j} - \vec{k})\,m$ and $\vec{E} = (4\,\vec{i} - 5\,\vec{j} + 8\,\vec{k})\,m$

Find the magnitude of the displacements $2\vec{D} - \vec{E}$

Example 18

Four forces act on bolt ***A*** as shown :



Determine the resultant of the forces on the bolt.

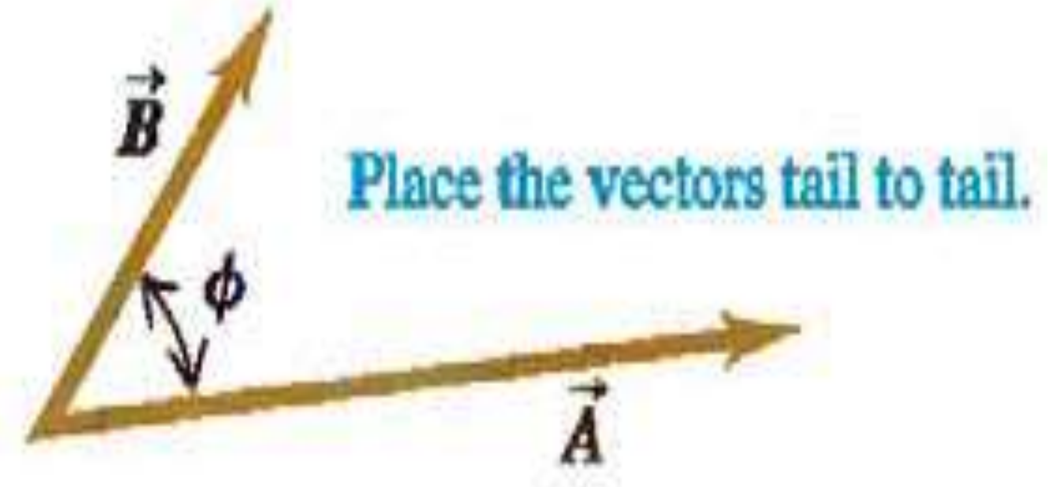
2.7. Products of Vectors

- Vectors are not ordinary numbers, so ordinary multiplication is not directly applicable to vectors.
- We will define two different kinds of products of vectors :
  - The first, called the scalar product, yields a result that is a scalar quantity.
  - The second, the vector product, yields another vector.

2.7.1. Scalar Product

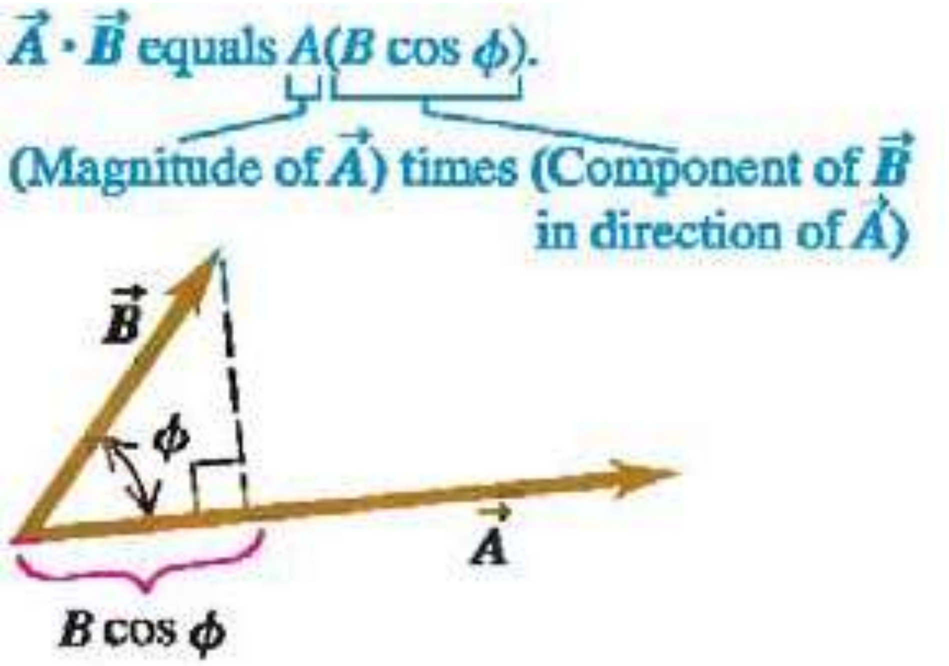
- The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}.\vec{B}$. Because of this notation, the scalar product is also called the dot product. Although $\vec{A}$ and $\vec{B}$ are vectors, the quantity $\vec{A}.\vec{B}$ is a scalar.

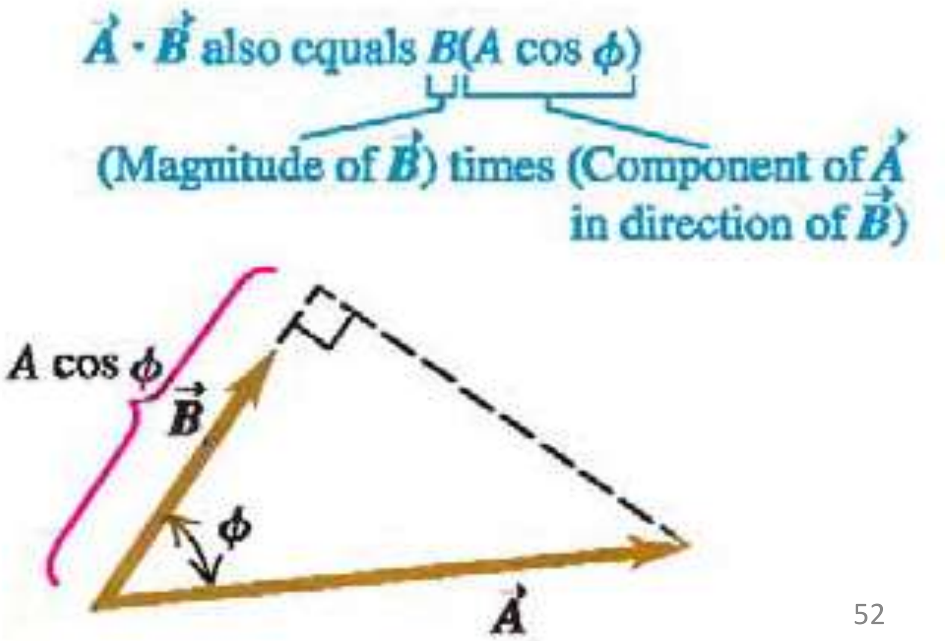
- To define the scalar product $\vec{A}.\vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$, we draw the two vectors with their tails at the same point. The angle ϕ between their directions ranges from $0°$ to $180°$.



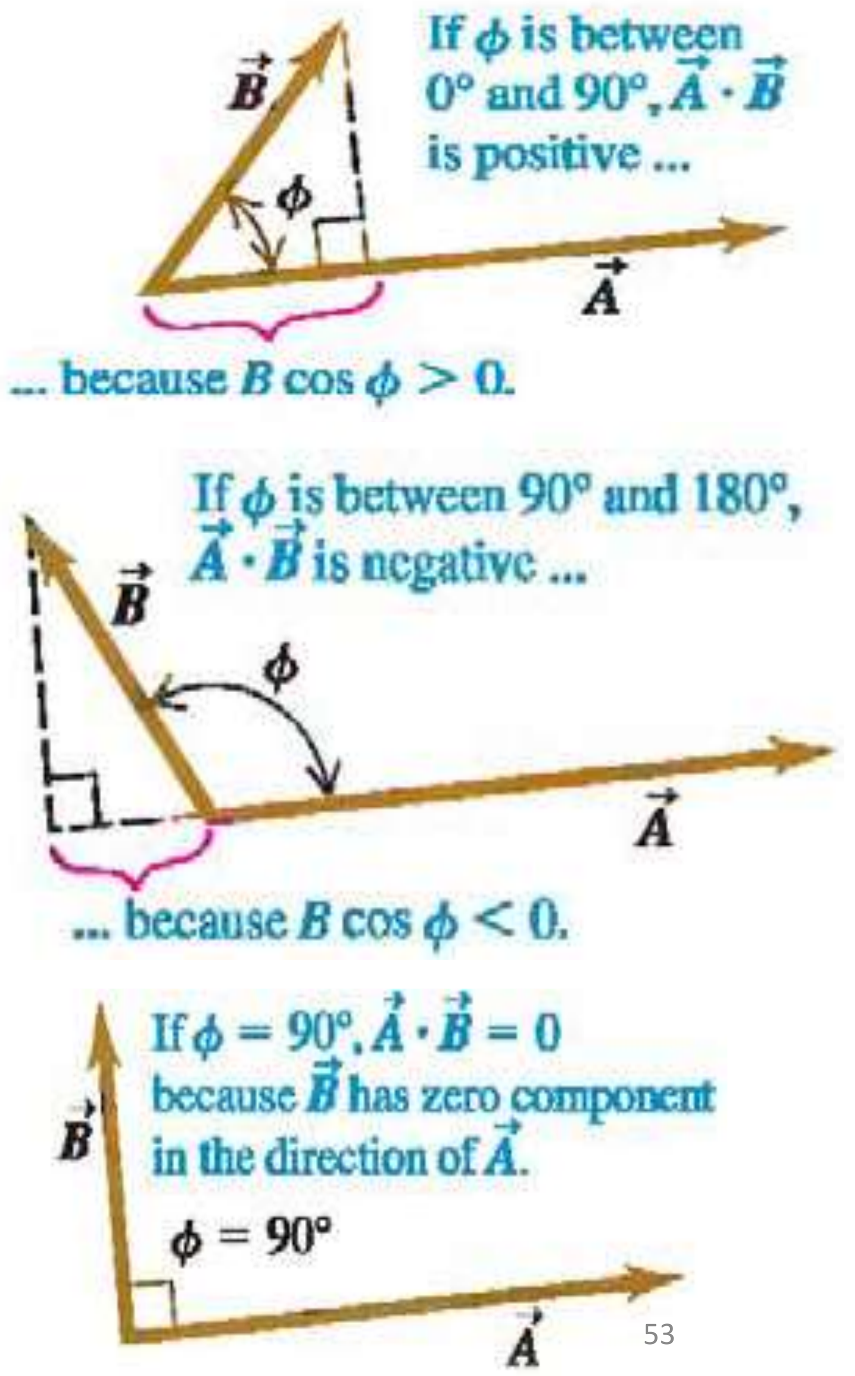
➢We define $\vec{A}.\vec{B}$ to be the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ in the direction of $\vec{A}$. Alternatively, we can define $\vec{A}.\vec{B}$ to be the magnitude of $\vec{B}$ multiplied by the component of $\vec{A}$ in the direction of $\vec{B}$. Expressed as an equation :

$$\boxed{\vec{A}.\vec{B} = \vec{B}.\vec{A} = A\,B\ \cos\phi = |\vec{A}|\,|\vec{B}|\ \cos\phi}$$





- The scalar product is a scalar quantity, not a vector, and it may be positive, negative, or zero.
- When ϕ is between $0°$ and $90°$, $\cos\phi > 0$ and the scalar product is positive.
- When ϕ is between $90°$ and $180°$ so that $\cos\phi < 0$, the component of $\vec{B}$ in the direction of $\vec{A}$ is negative, and $\vec{A}.\vec{B}$ is negative.
- Finally, when $\phi = 90°$, $\vec{A}.\vec{B} = 0$. The scalar product of two perpendicular vectors is always zero



Example 19

The vectors $\vec{A}$ and $\vec{B}$ are given by

$$\vec{A} = (2,3) \text{ and } \vec{B} = (-1,2)$$

a) Determine the scalar product $\vec{A}.\vec{B}$.

b) Find the angle ϕ between $\vec{A}$ and $\vec{B}$.

2.7.1.1. Calculating the Scalar Product Using Components

- We can calculate the scalar product $\vec{A}.\vec{B}$ directly if we know the x, y and z components of $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ and $\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$.
- To see how this is done, let's first work out the scalar product of the unit vectors. This is easy, since $\vec{i}$, $\vec{j}$, and $\vec{k}$ all have magnitude **1** and are perpendicular to each other. Using $\boxed{\vec{A}.\vec{B} = A\,B\,\cos\phi}$, we find

$$\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = (1)(1)\cos 0° = 1$$

$$\vec{i}.\vec{j} = \vec{i}.\vec{k} = \vec{j}.\vec{k} = (1)(1)\cos 90° = 0$$

- Now we express $\vec{A}.\vec{B}$ in terms of their components, expand the product, and use these products of unit vectors :

$$\vec{A}.\vec{B} = \left(A_x\vec{i} + A_y\vec{j} + A_z\vec{k}\right).\left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right)$$

$$\vec{A}.\vec{B} = A_x\vec{i}.B_x\vec{i} + A_x\vec{i}.B_y\vec{j} + A_x\vec{i}.B_z\vec{k} + A_y\vec{j}.B_x\vec{i} + A_y\vec{j}.B_y\vec{j} + A_y\vec{j}.B_z\vec{k} + A_z\vec{k}.B_x\vec{i} + A_z\vec{k}.B_y\vec{j} + A_z\vec{k}.B_z\vec{k}$$

$$\vec{A}.\vec{B} = A_x B_x \underbrace{\vec{i}.\vec{i}}_{1} + A_x B_y \underbrace{\vec{i}.\vec{j}}_{0} + A_x B_z \underbrace{\vec{i}.\vec{k}}_{0}$$

$$+A_y B_x \underbrace{\vec{j}.\vec{i}}_{0} + A_y B_y \underbrace{\vec{j}.\vec{j}}_{1} + A_y B_z \underbrace{\vec{j}.\vec{k}}_{0}$$

$$+ A_z B_x \underbrace{\vec{k}.\vec{i}}_{0} + A_z B_y \underbrace{\vec{k}.\vec{j}}_{0} + A_z B_z \underbrace{\vec{k}.\vec{k}}_{1}$$

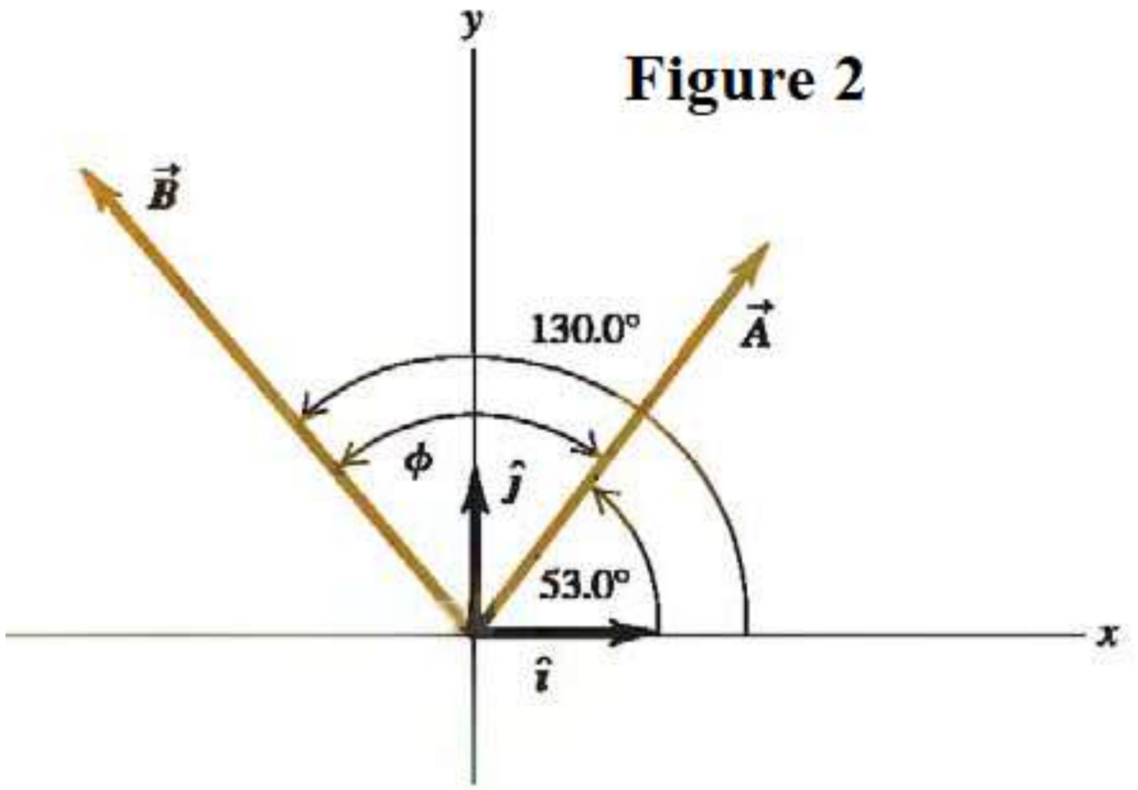
$$\boxed{\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z}$$

The scalar product of two vectors is the sum of the products of their respective components.

Example 20

Find by two methods the scalar product $\vec{A}.\vec{B}$ of the two vectors in Fig.2.

The magnitude of the vectors are $A = 4.00$ and $B = 5.00$



Example 21

Find the angle between the two vectors

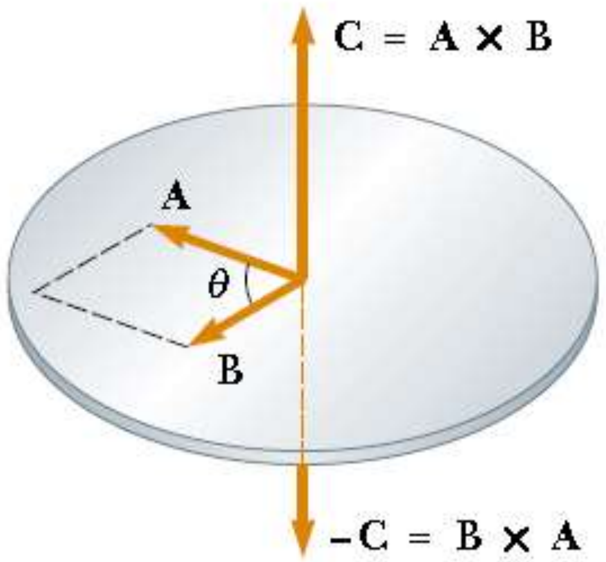
$\vec{A} = 2\,\vec{\imath} + 3\,\vec{\jmath} + 1\,\vec{k}$ and $\vec{B} = -4\,\vec{\imath} + 2\,\vec{\jmath} - 1\,\vec{k}$

2.7.2. Vector Product

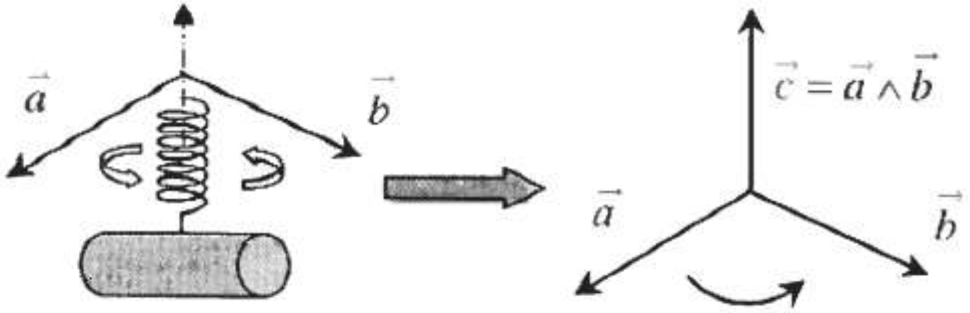
- Given any two vectors $\vec{A}$ and $\vec{B}$, the vector product $\vec{A} \times \vec{B}$ (or cross product) is defined as a third vector $\vec{C}$, which has a magnitude of $AB\ \sin\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$.

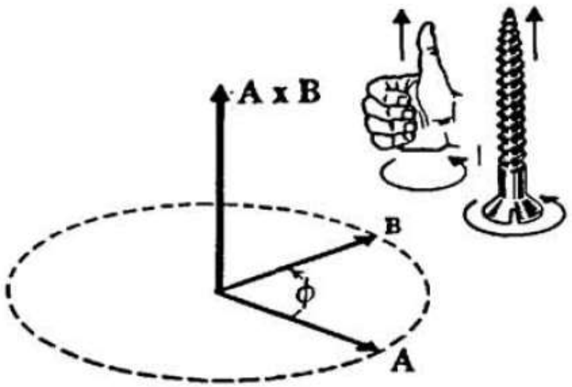
- That is, if $\vec{C}$ is given by : $\vec{C} = \vec{A} \wedge \vec{B}$, then its magnitude is : $\mathrm{C} = AB\ \sin\theta$

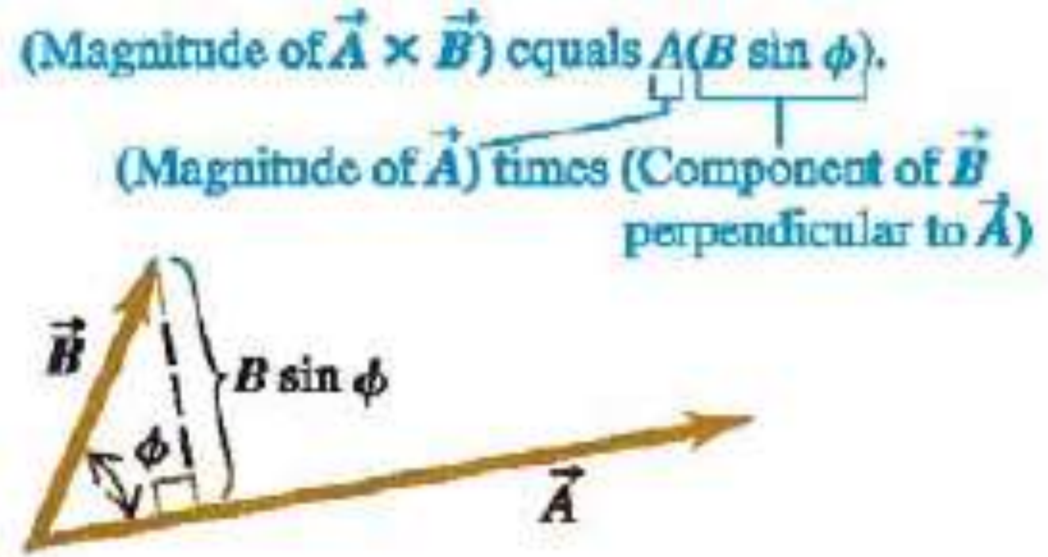
- The quantity $AB\ \sin\theta$ is equal to the area of the parallelogram formed by $\vec{A}$ and $\vec{B}$, as

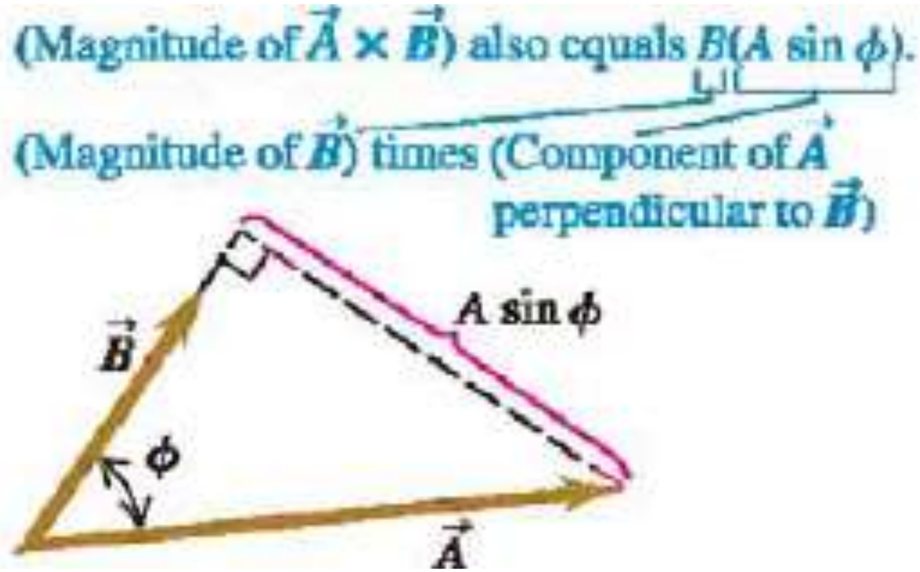


- Graphically, the cross product is constructed using the "Cork Screw rule". We have:









2.7.2.1. Properties of the vector product

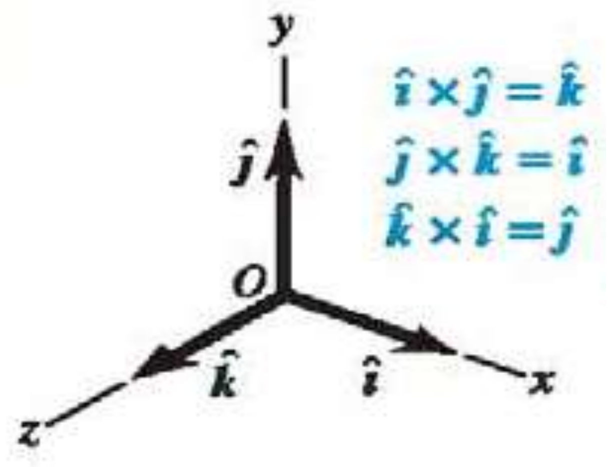
- The vector product is *not* commutative $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- If $\vec{A}$ is parallel to $\vec{B}$ $(\theta = 0° \; or \; 180°)$, then $\vec{A} \times \vec{B} = 0$; therefore, it follows that $\vec{A} \times \vec{A} = 0$.
- If $\vec{A}$ is perpendicular to $\vec{B}$, then $|\vec{A} \times \vec{B}| = AB$.
- The vector product obeys the distributive law : $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
- The derivative of the cross product with respect to some variable such as t is

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

where it is important to preserve the multiplicative order of $\vec{A}$ and $\vec{B}$

2.7.2.2. Calculating the Vector Product Using Components

- If we know the components of $\vec{A}$ and $\vec{B}$ ($\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ and $\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$) we can calculate the components of the vector product
- First, we work out the multiplication table for the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$, all three of which are perpendicular to each other.



- The vector product of any vector with it self is zero, so

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$$

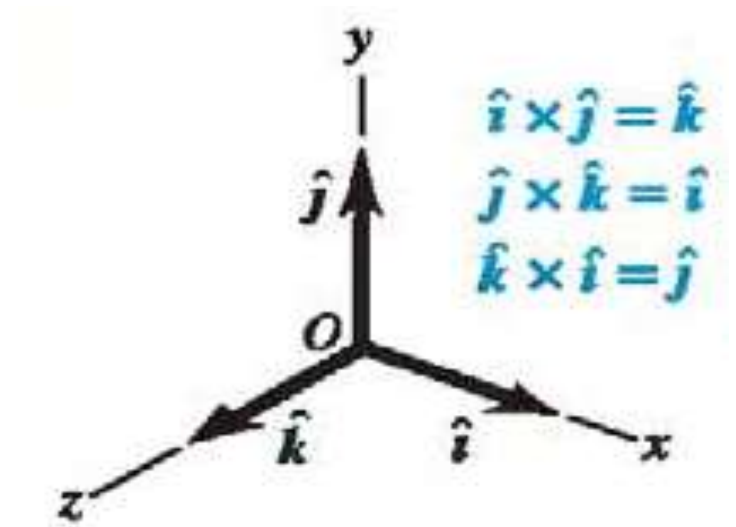
(a zero vector is one with all components equal to zero and an undefined direction)

➢ Using , $C = AB\sin\theta$; $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ and "Cork Screw rule", we find :

$$\vec{i} \times \vec{j} = -\vec{j} \times \vec{i} = \vec{k}$$

$$\vec{j} \times \vec{k} = -\vec{k} \times \vec{j} = \vec{i}$$

$$\vec{k} \times \vec{i} = -\vec{i} \times \vec{k} = \vec{j}$$



➢ $\vec{C} = \vec{A} \times \vec{B} = (A_x\vec{i} + A_y\vec{j} + A_z\vec{k}) \times (B_x\vec{i} + B_y\vec{j} + B_z k)$

$$\vec{C} = \vec{A} \times \vec{B} = A_x\vec{i} \times B_x\vec{i} + A_x\vec{i} \times B_y\vec{j} + A_x\vec{i} \times B_z\vec{k} +$$
$$A_y\vec{j} \times B_x\vec{i} + A_y\vec{j} \times B_y\vec{j} + A_y\vec{j} \times B_z\vec{k} +$$
$$A_z\vec{k} \times B_x\vec{i} + A_z\vec{k} \times B_y\vec{j} + A_z\vec{k} \times B_z\vec{k}$$

$$\vec{C} = \vec{A} \times \vec{B} = A_x B_x \underbrace{(\vec{i} \times \vec{i})}_{0} + A_x B_y \underbrace{(\vec{i} \times \vec{j})}_{\vec{k}} + A_x B_z \underbrace{(\vec{i} \times \vec{k})}_{-\vec{j}} +$$

$$A_y B_x \underbrace{(\vec{j} \times \vec{i})}_{-\vec{k}} + A_y B_y \underbrace{(\vec{j} \times \vec{j})}_{0} + A_y B_z \underbrace{(\vec{j} \times \vec{k})}_{\vec{i}} +$$

$$A_z B_x \underbrace{(\vec{k} \times \vec{i})}_{\vec{j}} + A_z B_y \underbrace{(\vec{k} \times \vec{j})}_{-\vec{i}} + A_z B_z \underbrace{(\vec{k} \times \vec{k})}_{0}$$

$$\boxed{\vec{C} = \vec{A} \times \vec{B} = \underbrace{\left(A_y B_z - A_z B_y\right)}_{C_x} \vec{i} + \underbrace{\left(A_z B_x - A_x B_z\right)}_{C_y} \vec{j} + \underbrace{\left(A_x B_y - A_y B_x\right)}_{C_z} \vec{k}}$$

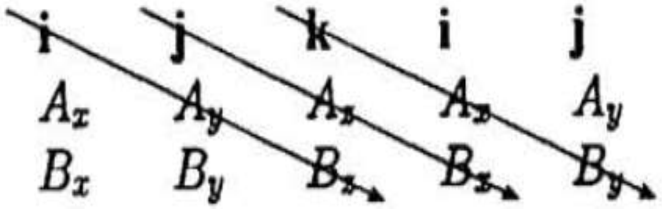
Thus, the components of $\vec{C} = \vec{A} \times \vec{B}$ are given by

$$C_x = A_y B_z - A_z B_y \quad ; \quad C_y = A_z B_x - A_x B_z \quad ; \quad C_z = A_x B_y - A_y B_x$$

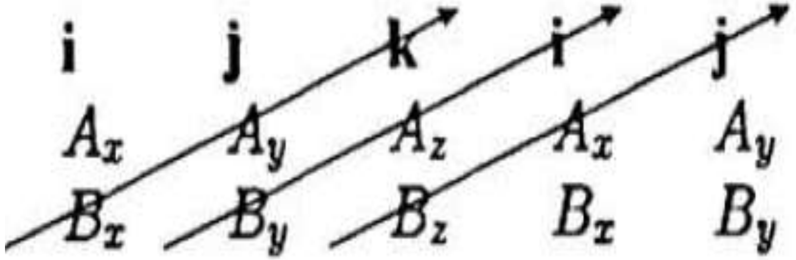
The vector product can also be expressed in determinant form as :

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\vec{i} + (A_zB_x - A_xB_z)\vec{j} + (A_xB_y - A_yB_x)\vec{k}$$

- To multiply this derminant, repeat the first two columns, as in figure.



- Imagine three downward sloping arrows, as shown.
- Multiply together the three elements on each arrow, and add them together to obtain $\overrightarrow{\boldsymbol{C_1}} = A_yB_z\,\vec{i} + A_zB_x\,\vec{j} + A_xB_y\,\vec{k}$
- Next, imagine three upward sloping arrows, as shown.

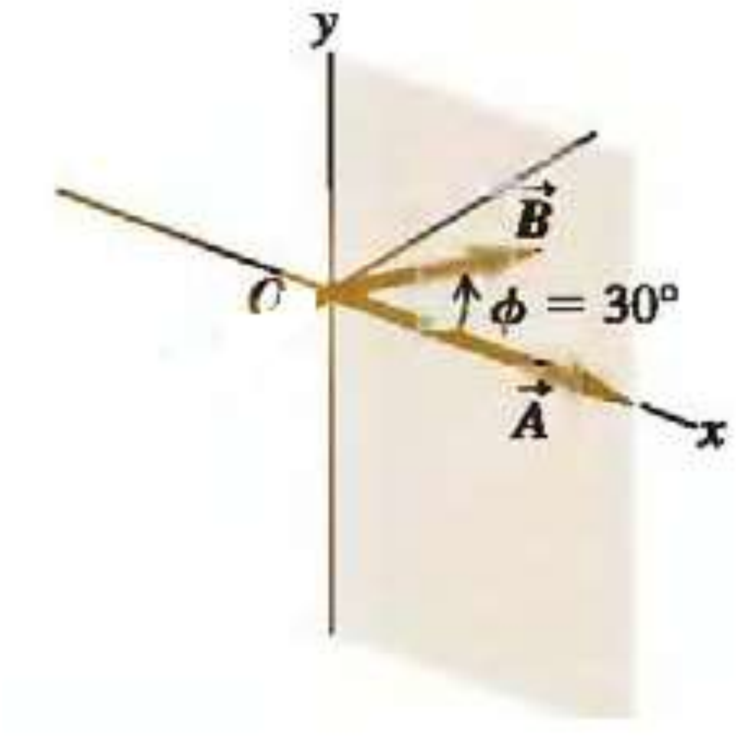


- Again multiply together the three elements on each arrow, obtaining $\overrightarrow{\boldsymbol{C_2}} = A_yB_x\,\vec{k} + A_zB_y\,\vec{i} + A_xB_z\,\vec{j}$
- Substract $\overrightarrow{\boldsymbol{C_2}}$ from $\overrightarrow{\boldsymbol{C_1}}$ to obtain $\vec{C} = \overrightarrow{\boldsymbol{C_1}} - \overrightarrow{\boldsymbol{C_2}} = \vec{A} \times \vec{B}$

$$\boxed{\vec{C} = \overrightarrow{\boldsymbol{C_1}} - \overrightarrow{\boldsymbol{C_2}} = \vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\vec{i} + (A_zB_x - A_xB_z)\vec{j} + (A_xB_y - A_yB_x)\vec{k}}$$

Example 22

Vector $\vec{A}$ has magnitude 6 units and is in the direction of the $+x$-axis. Vector $\vec{B}$ has magnitude 4 units and lies in the xy-plane, making an angle of 30° with the $+x$-axis



Find the vector product $\vec{A} \times \vec{B}$

Example 23

a) The vector $(1, a, b)$ is perpendicular to both vectors $(4, 3, 0)$ and $(5, 1, 7)$. Find a and b.

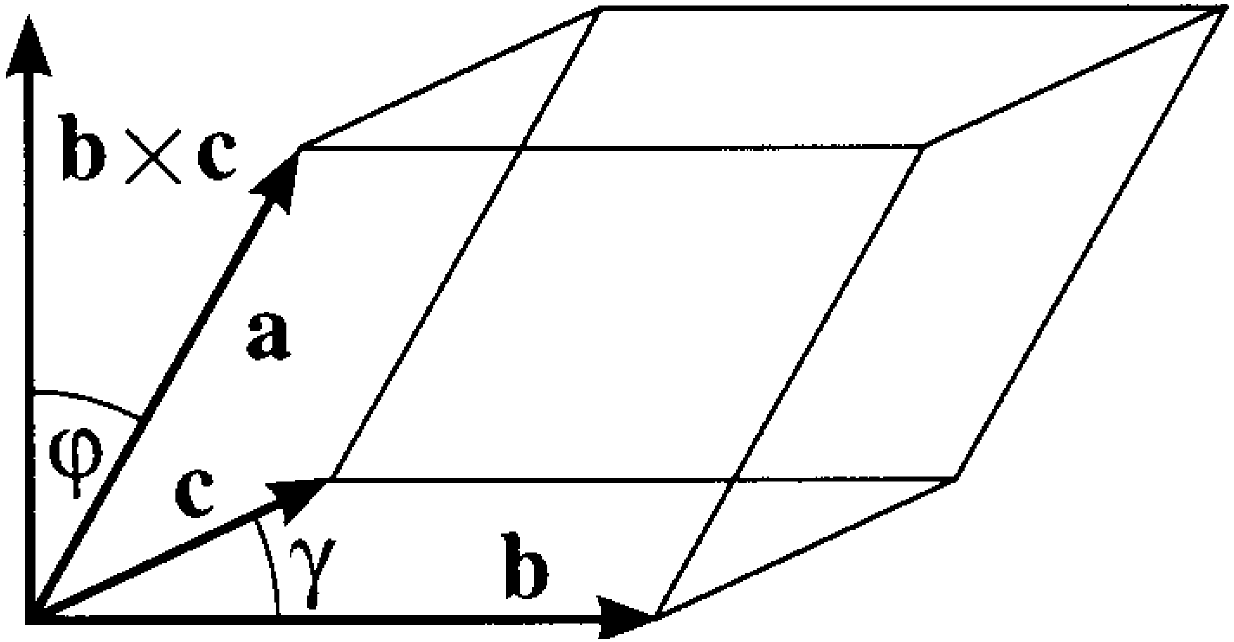
b) For any two vectors, show that: $\left(\vec{A} \times \vec{B}\right)^2 = A^2\,B^2 - \left(\vec{A}.\vec{B}\right)^2$

2.8.The triple scalar product

- The triple scalar product of the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $\vec{a}.(\vec{b} \times \vec{c})$,
- that is, a combination of a scalar and vector product.
- The triple scalar product is therefore also denoted as a mixed product. The triple scalar product is a scalar.
- $\vec{a}.(\vec{b} \times \vec{c}) = (a_x, a_y, a_z).[(b_x, b_y, b_z) \times (c_x, c_y, c_z)]$

$$\vec{a}.(\vec{b} \times \vec{c}) = (a_x, a_y, a_z).\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = (a_x, a_y, a_z).[(A_y B_z - A_z B_y)\vec{i}$$

- Geometrically, the triple scalar product represents the volume of a parallelepipedon formed by the three vectors (see figure).



$$V = \vec{a}.\left(\vec{b} \times \vec{c}\right) = a\cos\varphi\, bc\sin\gamma$$

$$= a\, bc\cos\varphi\sin\gamma$$

Example 24

Find the volume V of a parallelepiped with sides

$$\vec{A} = 3\,\vec{i} - 1\,\vec{j} \;\; ; \quad \vec{B} = \vec{j} + 2\,\vec{k} \;\; ; \quad \vec{C} = 1\,\vec{i} + 5\,\vec{j} + 4\,\vec{k}$$

2.9. Gradient, Divergence, and Rotation

- If at each point (x, y, z) of a Cartesian coordinate system, we associate a vector $\vec{A}$, we say that $\vec{A} = \vec{A}(x, y, z)$ is a vector function of x, y, z; we then call $\vec{A}(x, y, z)$ a vector field.

- Similarly, we call the scalar function $\varphi(x, y, z)$ a scalar field.

- It is convenient to introduce the differential vector operator $\vec{\nabla}$, called nabla,

defined by $$\boxed{\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}}$$

- Using this, we define the following quantities:

2.9.1 Gradient

$$\vec{\nabla}\phi = \left(\vec{i}\,\frac{\delta}{\delta x} + \vec{j}\,\frac{\delta}{\delta y} + \vec{k}\,\frac{\delta}{\delta z}\right)\phi$$

$$\vec{\nabla}\phi \; = \vec{i}\,\frac{\delta\phi}{\delta x} + \vec{j}\,\frac{\delta\phi}{\delta y} + \vec{k}\,\frac{\delta\phi}{\delta z}$$

This is a vector called the gradient of ϕ and can also be written as $(grad\ \phi)$

$$\boxed{\vec{\nabla}\phi = grad\ \phi = \vec{i}\,\frac{\delta\phi}{\delta x} + \vec{j}\,\frac{\delta\phi}{\delta y} + \vec{k}\,\frac{\delta\phi}{\delta z}}$$

2.9.2. Divergence

$$\vec{\nabla}.\vec{A} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).\left(A_x\,\vec{i} + A_y\,\vec{j} + A_z\,\vec{k}\right)$$

$$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

it is a scalar called divergence of $\vec{A}$ and which is also written $(div\vec{A})$.

$$\boxed{\vec{\nabla}.\vec{A} = div\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

2.9.3. Rotation (or Curl)

$$\vec{\nabla} \times \vec{A} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \times \left(A_x\,\vec{i} + A_y\,\vec{j} + A_z\,\vec{k}\right)$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k}$$

This is a vector called the rotation (or curl) of $\vec{A}$ and which can also be written as $(rot\ \vec{A})$ or $(curl\ \vec{A})$

$$\boxed{\vec{\nabla} \times \vec{A} = rot\ \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k}}$$

Example 25

If $\phi = x^2yz^3$ and $\vec{A} = xz\,\vec{\imath} - y^2\,\vec{\jmath} + 2x^2y\,\vec{k}$, find :

a) $\vec{\nabla}\phi$

b) $\vec{\nabla}.\vec{A}$

c) $\vec{\nabla} \wedge \vec{A}$

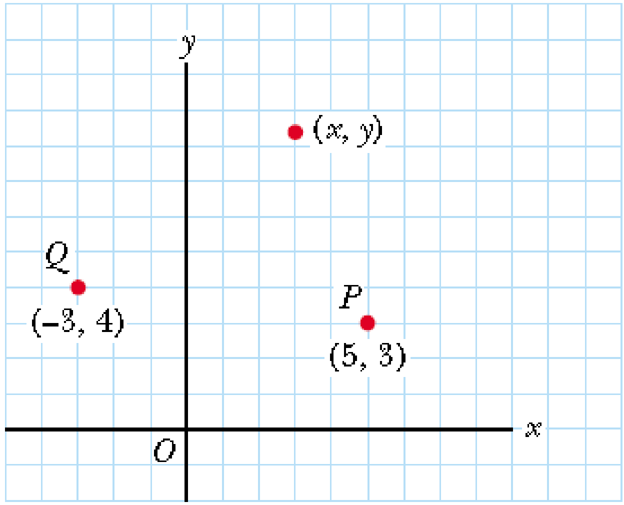
d) $div\,(\phi\vec{A})$

e) $rot\,(\phi\vec{A})$

3. Coordinate Systems

- Many aspects of physics involve a description of a location in space.
- In Chapter 2, for example, we will see that the mathematical description of an object's motion requires a method for describing the object's position at various times.
- This description is accomplished with the use of coordinates.
- Various coordinate systems can be used.

3.1. Cartesian coordinate system

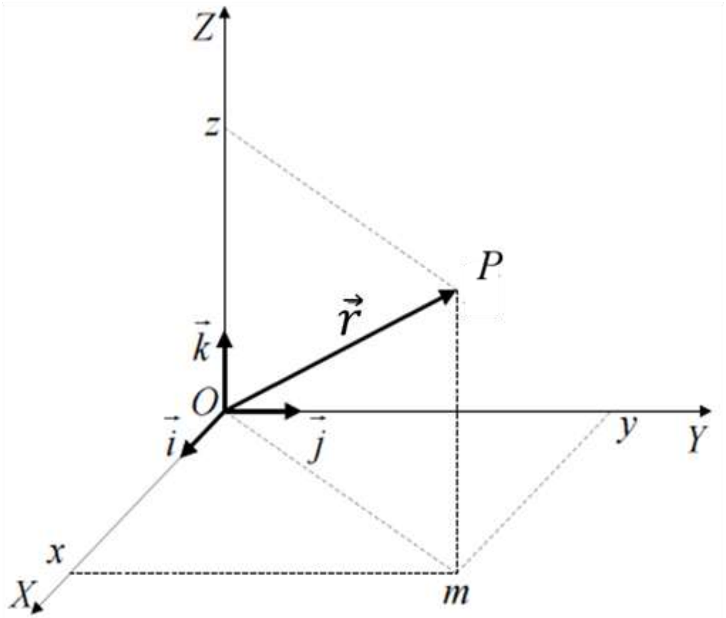


- Cartesian coordinate system, in which horizontal and vertical axes intersect at a point defined as the origin. Cartesian coordinates are also called rectangular coordinates.

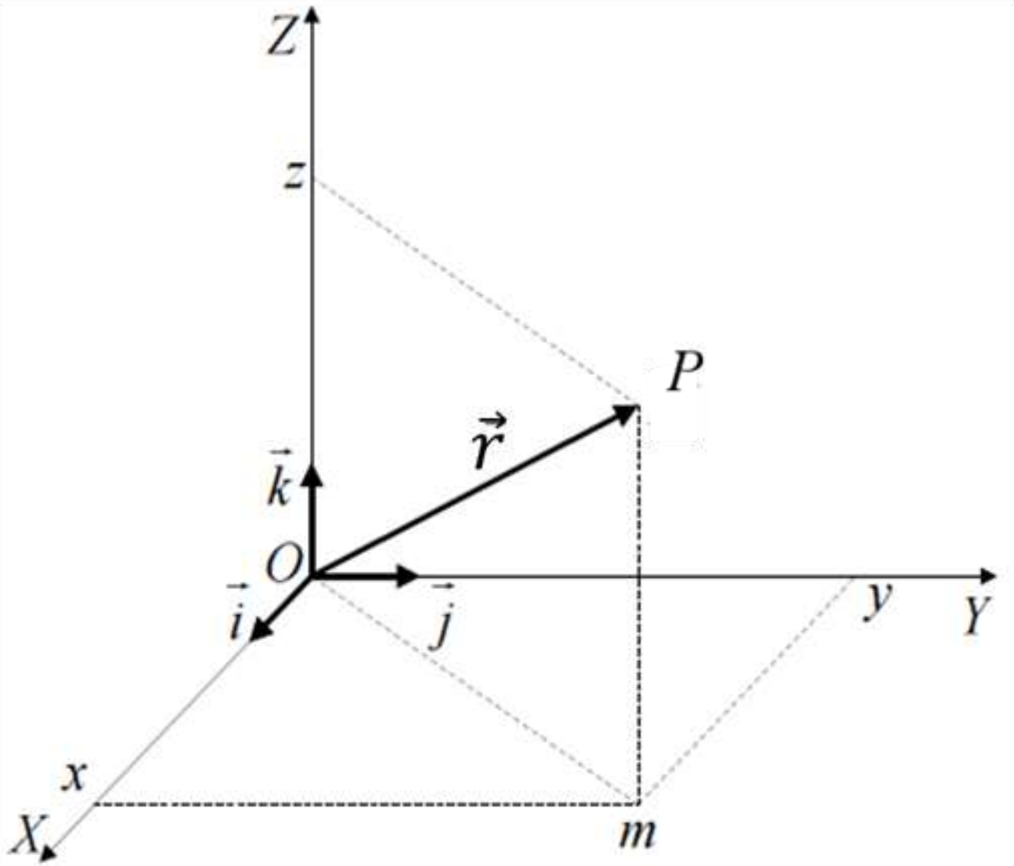
- The basis $(\vec{i}, \vec{j}, \vec{k})$ is orthonormal and direct, which implies the following properties:

$$\text{Orthonormal} \Leftrightarrow \begin{cases} \vec{i}.\vec{j} = 0 \\ \vec{i}.\vec{k} = 0 \\ \vec{j}.\vec{k} = 0 \end{cases} \text{ and } \{\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1\}$$

$$\text{Direct} \Leftrightarrow \begin{cases} \vec{i} \wedge \vec{j} = \vec{k} \\ \vec{j} \wedge \vec{k} = \vec{i} \\ \vec{k} \wedge \vec{i} = \vec{j} \end{cases}$$



- The coordinate lines are straight lines parallel to the coordinate axes. To describe all space, x, y and z must vary from $-\infty$ to $+\infty$.
- The Cartesian basis $(\vec{i}, \vec{j}, \vec{k})$ is invariant : $\frac{d\vec{i}}{dt} = \frac{d\vec{j}}{dt} = \frac{d\vec{k}}{dt} = 0$
- Position vector : $\vec{r} = \overrightarrow{OP} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$.

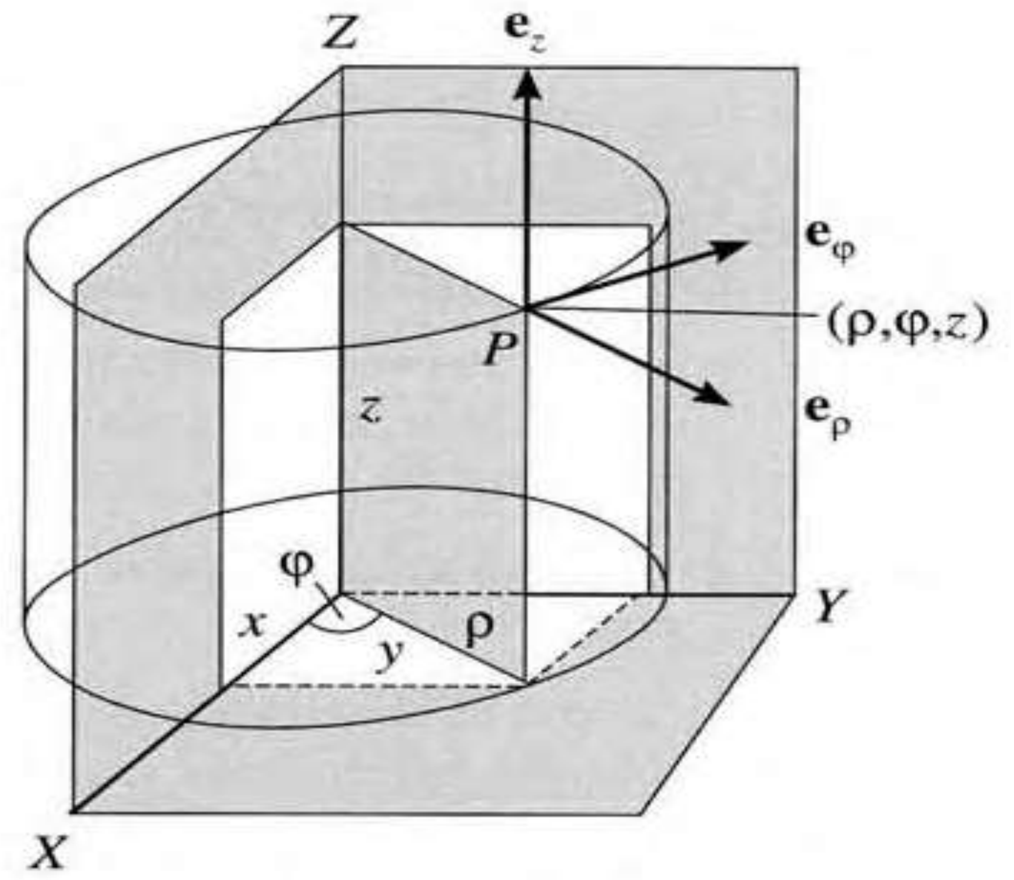


3.2. The cylindrical and polar coordinates system

ρ : polar radius ; φ : polar angle; z : altitude

- When a problem has a privileged axis, such as an axis of rotation, or an axis of symmetry, it is preferable to use a coordinate system that coincides with this particular geometry: this is the cylindrical coordinate system.
- In this system, point P is located using its cylindrical coordinates ρ, φ , z and a basis $\left(\overrightarrow{e_\rho}, \overrightarrow{e_\varphi}, \overrightarrow{e_z} = \vec{k}\right)$

➢The coordinate line on ρ is the radial half-line passing through P, the one along φ is a circle of radius ρ located in a plane orthogonal to (Oz), and the one along z is the line parallel to (Oz) passing through P.



➢ To describe the entire space,

ρ must vary from 0 to $+\infty$,

φ from 0 to 2π, and z from $-\infty$ to $+\infty$.

➢In the cylindrical basis, the radial unit $\overrightarrow{e_\rho}$ and the orthoradial unit $\overrightarrow{e_\varphi}$ vary with the position of P (they depend on φ). The axial unit $\overrightarrow{e_z}$ is invariant.

➢Position vector: $\overrightarrow{OP} = \rho\overrightarrow{e_\rho} + z\overrightarrow{e_z}$ Note that the orthoradial unit $\overrightarrow{e_\varphi}$ does not appear in this expression.

Note:

- If the point must be identified in the plan, it is possible to use polar coordinates.
- Simply remove the Oz axis with its coordinate from the cylindrical coordinate system to obtain the polar coordinates.

Example 26

1) Determine the expressions for the Cartesian coordinates x and y in terms of the polar coordinates (r, θ).

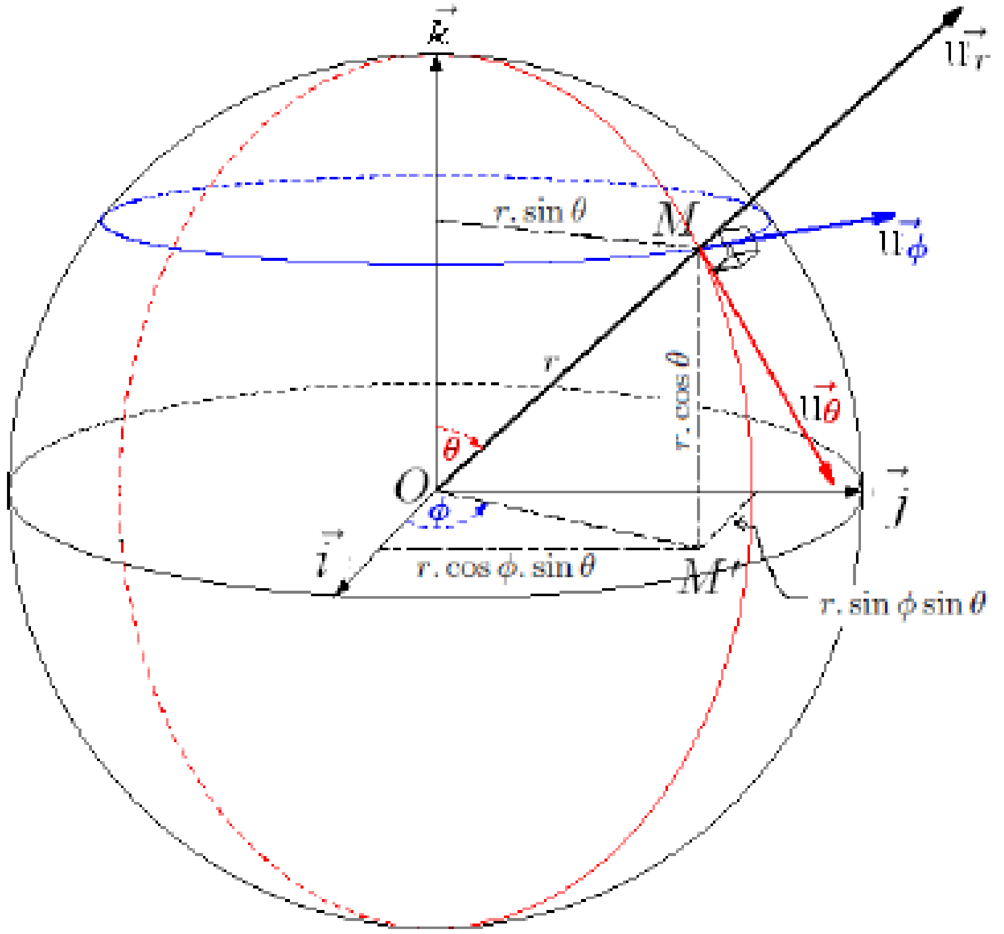
2) Given the expressions for $(\overrightarrow{e_r}, \overrightarrow{e_\theta})$, the polar unit vectors, in terms of $(\vec{i}, \vec{j})$, the Cartesian unit vectors.

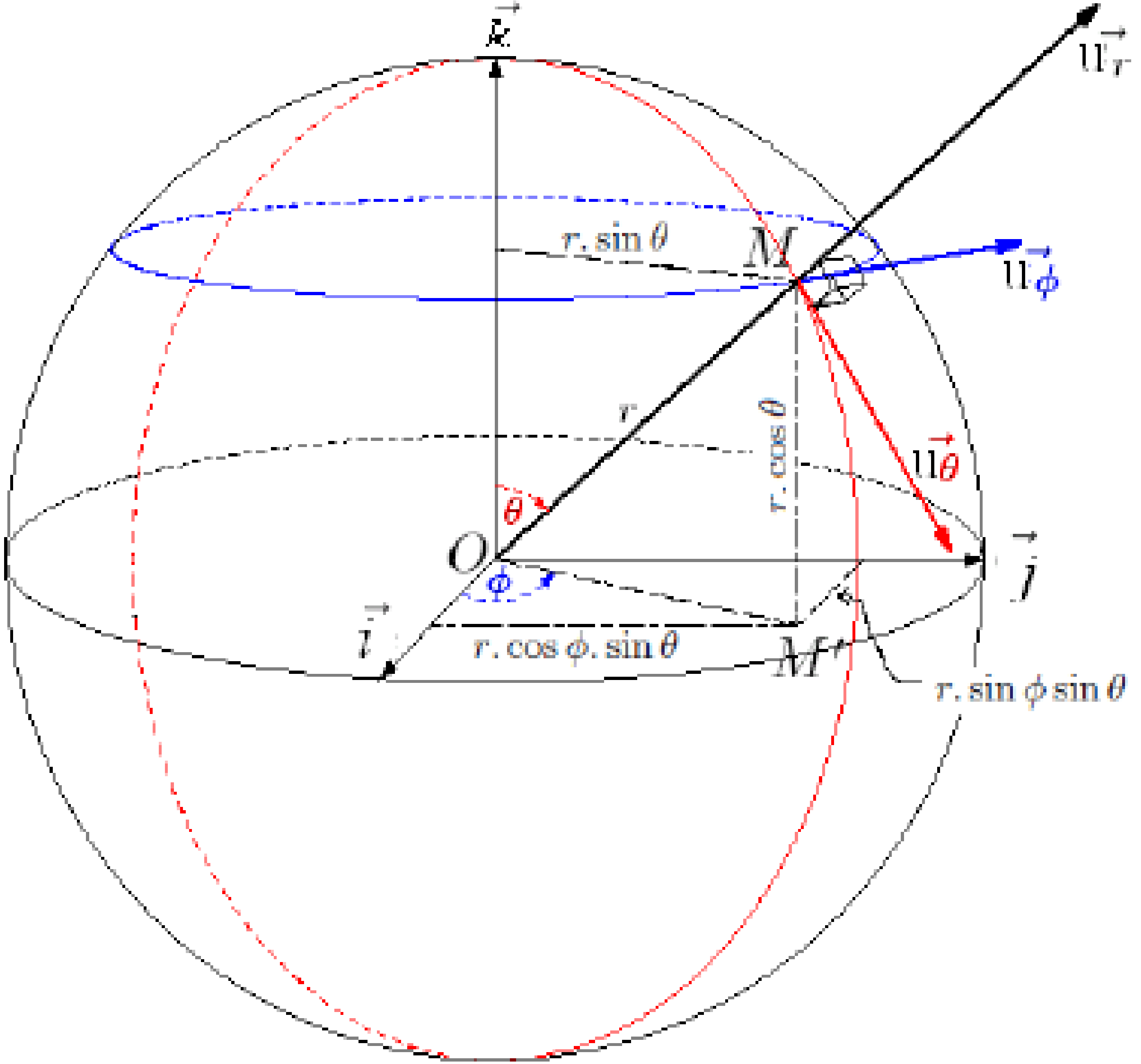
3) Calculate: $\frac{d\overrightarrow{e_r}}{d\theta}$ and $\frac{d\overrightarrow{e_r}}{dt}$; $\frac{d\overrightarrow{e_\theta}}{d\theta}$ and $\frac{d\overrightarrow{e_\theta}}{dt}$

3.3. The spherical coordinate system

r: Polar radius; θ : Colatitude; φ : Azimuth or longitude

This coordinate system will be used when the properties in the problem to be solved depend only on the distance between the center of the coordinate system and the point M considered (no direction is favored and the properties are said to be radial)




$\vec{k}$
$\vec{u}_r$
$r.\sin\theta$
M
$\vec{u}_\phi$
r
$r.\cos\theta$
$\vec{u}_\theta$
θ
O
$\vec{j}$
ϕ
$\vec{i}$
$r.\cos\phi.\sin\theta$
M'
$r.\sin\phi\sin\theta$

- The coordinate line along r is the spherical radial half-line (OM);
- the one along θ is the circle centered at O with radius r lying in the plane $(O, M, \acute{M})$. (where $\acute{M}$ is the orthogonal projection of M onto the plane (Oxy));
- the one along φ is the circle with axis (Oz) passing through M, with radius $\rho = r\sin\theta$.
- To span the entire space, r must range from 0 to $+\infty$, θ from 0 to π and φ from 0 to 2π.
- In the spherical basis, the radial unit vector $\overrightarrow{u_r}$ and the orthoradial unit vector $\overrightarrow{u_\theta}$ vary with the position of M (they depend on both θ and φ). The azimuthal unit vector $\overrightarrow{u_\varphi}$ also varies with M, as it depends on φ.
- Position vector: $\overrightarrow{\boldsymbol{OM}} = r\,\overrightarrow{u_r}$. Note that neither the orthoradial unit vector nor the azimuthal unit vector appear in this expression.

Example 27

1) Determine the expressions for the Cartesian coordinates x, y and z as functions of the spherical coordinates (r, θ, φ).

2) Give the expressions for the spherical unit vectors $(\overrightarrow{u_r}, \overrightarrow{u_\theta}, \overrightarrow{u_\varphi})$ in terms of the Cartesian unit vectors $(\vec{i}, \vec{j}, \vec{k})$.

3) Compute: $\frac{d\overrightarrow{u_r}}{dt}$, $\frac{d\overrightarrow{u_\varphi}}{dt}$ and $\frac{d\overrightarrow{u_\theta}}{dt}$