

Chapter I: Vector analysis

Introduction

The concept of a vector space is a fundamental structure in modern mathematics. The aim is to identify the common properties shared by very different sets. For example, one can add two vectors in the plane, and also multiply a vector by a real number (to stretch or shrink it). Similarly, one can add two functions, or multiply a function by a real number. The same applies to polynomials, matrices, etc. The goal is to obtain general theorems that apply equally to vectors in the plane or space, function spaces, polynomials, matrices, etc. The counterpart to this great generality is that the concept of a vector space is difficult to grasp and will require a significant amount of work! It is advisable to first study the chapter “The Vector Space $\mathbb{R}^n$ ”.

1 Vector Space

In this chapter, K denotes a field. In most examples, it will be the field of real numbers $\mathbb{R}$.

1.1 Definition of a Vector Space

A vector space is a set consisting of vectors, such that one can add (and subtract) two vectors u, v to form a third $u + v$ (or $u - v$), and also multiply each vector u by a scalar λ to obtain a vector $\lambda \cdot u$. Here is the formal definition:

Definition A K -vector space is a non-empty set E equipped with:

- an internal composition law, i.e., a mapping from $E \times E$ to E :

$$\begin{aligned} E \times E &\rightarrow E \\ (u, v) &\mapsto u + v \end{aligned}$$

- an external composition law, i.e., a mapping from $K \times E$ to E :

$$\begin{aligned} K \times E &\rightarrow E \\ (\lambda, u) &\mapsto \lambda \cdot u \end{aligned}$$

satisfying the following properties:

1. $u + v = v + u$ for all $u, v \in E$.
2. $u + (v + w) = (u + v) + w$ for all $u, v, w \in E$.

3. There exists a neutral element $0_E \in E$ such that $u + 0_E = u$ for all $u \in E$.
4. Every $u \in E$ has a symmetric element u' such that $u + u' = 0_E$. This element u' is denoted $-u$.
5. $1 \cdot u = u$ for all $u \in E$.
6. $\lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u$ for all $\lambda, \mu \in K, u \in E$.
7. $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$ for all $\lambda \in K, u, v \in E$.
8. $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ for all $\lambda, \mu \in K, u \in E$.

We will return to each of these properties in detail after some examples.

Example Let $K = \mathbb{R}$ and $E = \mathbb{R}^2$. An element $u \in E$ is a pair (x, y) with $x \in \mathbb{R}$ and $y \in \mathbb{R}$. This is written as:

$$\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}.$$

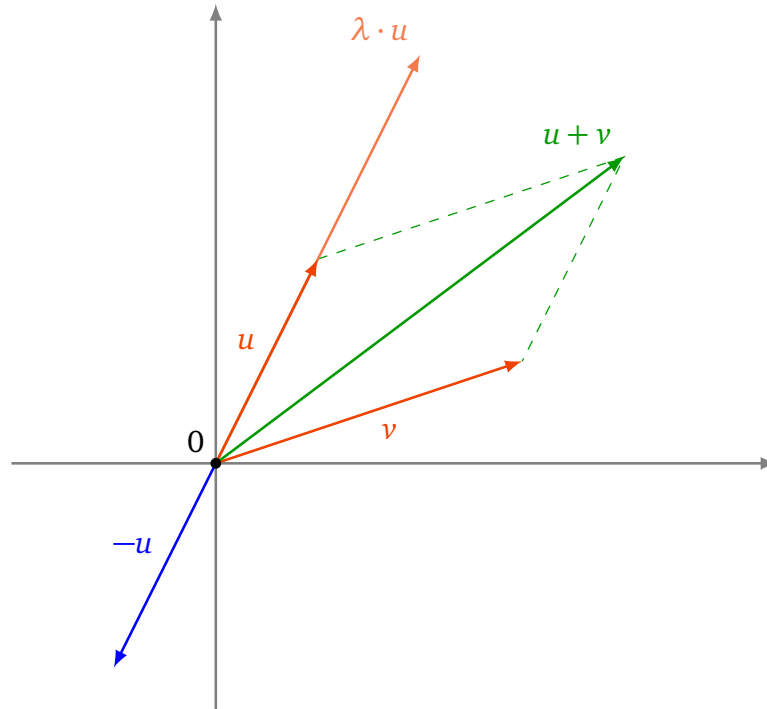
- Definition of the internal law. If (x, y) and (x', y') are two elements of $\mathbb{R}^2$, then:

$$(x, y) + (x', y') = (x + x', y + y').$$

- Definition of the external law. If λ is a real number and (x, y) is an element of $\mathbb{R}^2$, then:

$$\lambda \cdot (x, y) = (\lambda x, \lambda y).$$

The neutral element of the internal law is the zero vector $(0, 0)$. The symmetric of (x, y) is $(-x, -y)$, also denoted $-(x, y)$.



1.2 Terminology and Notation

Let us gather the definitions already seen.

The elements of E are called *vectors*. Instead of K -vector space, we also say vector space over K . The elements of K are called *scalars*. The neutral element 0_E is also called the *zero vector*. It must not be confused with the element 0 of K . When there is no risk of confusion, 0_E will also be denoted 0 . The symmetric element $-u$ of a vector $u \in E$ is also called the *opposite*. The internal composition law on E (usually denoted $+$) is commonly called *addition*, and $u + u'$ is called the *sum* of vectors u and u' . The external composition law on E is commonly called *scalar multiplication*. The multiplication of vector u by scalar λ is often written simply as λu , instead of $\lambda \cdot u$.

Sum of n vectors. It is possible to define, by induction, the addition of n vectors for $n \geq 2$. The vector space structure allows us to define the addition of two vectors (initializing the process). If the sum of $n - 1$ vectors is defined, then the sum of n vectors $v_1, v_2, \dots, v_n$ is defined by:

$$v_1 + v_2 + \dots + v_n = (v_1 + v_2 + \dots + v_{n-1}) + v_n.$$

The associativity of the $+$ law allows us to omit parentheses in the sum $v_1 + v_2 + \dots + v_n$. We denote:

$$v_1 + v_2 + \dots + v_n = \sum_{i=1}^n v_i.$$

1.3 Calculation Rules

Proposition *Let E be a vector space over a field K . Let $u \in E$ and $\lambda \in K$. Then:*

1. $0 \cdot u = 0_E$
2. $\lambda \cdot 0_E = 0_E$
3. $(-1) \cdot u = -u$
4. $\lambda \cdot u = 0_E \iff \lambda = 0 \text{ or } u = 0_E$

The operation that assigns $u + (-v)$ to (u, v) is called *subtraction*. The vector $u + (-v)$ is denoted $u - v$. The following properties hold: $\lambda(u - v) = \lambda u - \lambda v$ and $(\lambda - \mu)u = \lambda u - \mu u$.

2 Basis

2.1 Linear Combination

Let E be a K -vector space.

Definition *Let $v_1, v_2, \dots, v_p$ (with $p \geq 1$) be vectors in a vector space E . Any vector of the form*

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_p v_p$$

(where $\lambda_1, \lambda_2, \dots, \lambda_p \in K$) is called a linear combination of the vectors $v_1, v_2, \dots, v_p$. The scalars $\lambda_1, \lambda_2, \dots, \lambda_p$ are called the coefficients of the linear combination.

2.2 Linearly Independent Family

Definition A family $\{v_1, v_2, \dots, v_p\}$ of vectors in E is called a linearly independent family (or simply independent) if every linear combination that equals the zero vector

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_p v_p = 0$$

has all its coefficients zero, i.e.,

$$\lambda_1 = 0, \lambda_2 = 0, \dots, \lambda_p = 0.$$

Otherwise, i.e., if there exists a zero linear combination with not all coefficients zero, the family is called linearly dependent. Such a linear combination is then called a linear dependence relation among the v_j .

For vectors in $\mathbb{R}^n$, deciding whether a family $\{v_1, \dots, v_p\}$ is independent or dependent reduces to solving a linear system.

Example In the $\mathbb{R}$ -vector space $\mathbb{R}^3$, consider the family

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}.$$

We wish to determine whether it is independent or dependent. We look for scalars $(\lambda_1, \lambda_2, \lambda_3)$ such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0,$$

which is equivalent to the system:

$$\begin{cases} \lambda_1 + 4\lambda_2 + 7\lambda_3 = 0 \\ 2\lambda_1 + 5\lambda_2 + 8\lambda_3 = 0 \\ 3\lambda_1 + 6\lambda_2 + 9\lambda_3 = 0 \end{cases}$$

We compute that this system is equivalent to:

$$\begin{cases} \lambda_1 + 4\lambda_2 + 7\lambda_3 = 0 \\ -3\lambda_2 - 6\lambda_3 = 0 \end{cases}$$

This system has infinitely many solutions. Taking for example $\lambda_3 = 1$, we get $\lambda_1 = 1$ and $\lambda_2 = -2$, giving:

$$2v_1 - v_2 + v_3 = 0.$$

Thus the family is linearly dependent.

Example Consider the family:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We look for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0.$$

This leads to the system:

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}$$

Solving this system gives the unique solution $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$. Hence the family is linearly independent.

Example Let

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Then $\{v_1, v_2, v_3\}$ is linearly dependent because

$$3v_1 + v_2 - v_3 = 0.$$

2.3 Linearly Dependent Family

Let E be a K -vector space. If $v \neq 0$, the single-vector family $\{v\}$ is linearly independent (and dependent if $v = 0$). Consider the special case of a family of two vectors.

Proposition The family $\{v_1, v_2\}$ is linearly dependent if and only if v_1 is a scalar multiple of v_2 or v_2 is a scalar multiple of v_1 .

By contraposition: “The family $\{v_1, v_2\}$ is linearly independent if and only if v_1 is not a multiple of v_2 and v_2 is not a multiple of v_1 .”

2.4 Generating Family (Spanning Set)

Let E be a vector space over a field K .

Definition Let $v_1, \dots, v_p$ be vectors in E . The family $\{v_1, \dots, v_p\}$ is called a generating family (or spanning set) of the vector space E if every vector in E is a linear combination of $v_1, \dots, v_p$. This can be written as:

$$\forall v \in E, \exists \lambda_1, \dots, \lambda_p \in K \quad v = \lambda_1 v_1 + \dots + \lambda_p v_p.$$

We also say that the family $\{v_1, \dots, v_p\}$ spans the vector space E .

This notion is of course linked to the notion of the span of vectors: $\{v_1, \dots, v_p\}$ forms a generating family of E if and only if $E = \text{Vect}(v_1, \dots, v_p)$.

Example Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

in $E = \mathbb{R}^3$. The family $\{v_1, v_2, v_3\}$ is generating because any vector

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

can be written as

$$v = xv_1 + yv_2 + zv_3.$$

Here the coefficients are $\lambda_1 = x, \lambda_2 = y, \lambda_3 = z$.

Example Now consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

in $E = \mathbb{R}^3$. The family $\{v_1, v_2\}$ does not span $\mathbb{R}^3$. For instance, the vector

$$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is not in $\text{Vect}(v_1, v_2)$. Indeed, if it were, there would exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $v = \lambda_1 v_1 + \lambda_2 v_2$, i.e.,

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_1 = 0 \\ \lambda_2 = 1 \end{cases}$$

This system has no solution (the first two equations give $\lambda_1 = \lambda_2 = 0$, which contradicts the third).

Example Let $E = \mathbb{R}^2$. Let

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The family $\{v_1, v_2\}$ spans $\mathbb{R}^2$ because any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can be written as

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now let

$$v'_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v'_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then $\{v'_1, v'_2\}$ also spans $\mathbb{R}^2$. Indeed, for any $v = \begin{pmatrix} x \\ y \end{pmatrix}$, we look for $\lambda, \mu \in \mathbb{R}$ such that $v = \lambda v'_1 + \mu v'_2$, i.e.:

$$\begin{cases} 2\lambda + \mu = x \\ \lambda + \mu = y \end{cases}$$

The solution is $\lambda = x - y$ and $\mu = -x + 2y$, which exists for all real x, y . This shows that there can be several different finite families, not contained in one another, that span the same vector space.

2.5 Connections Between Generating Families

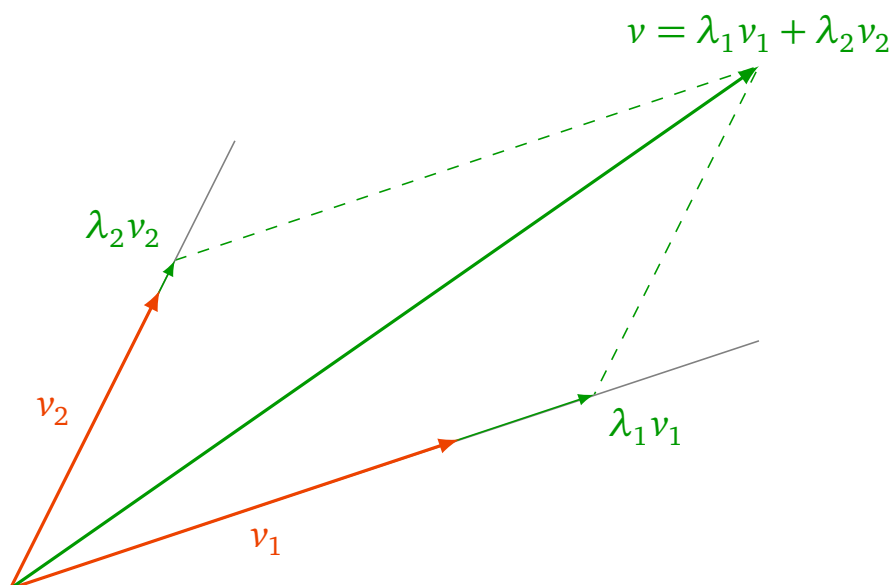
The following proposition is often useful:

Proposition Let $F = \{v_1, v_2, \dots, v_p\}$ be a generating family of E . Then $F' = \{v'_1, v'_2, \dots, v'_q\}$ is also a generating family of E if and only if every vector in F is a linear combination of vectors in F' .

2.6 Basis

The concept of a basis generalizes the concept of a coordinate system. In $\mathbb{R}^2$, a coordinate system is given by a pair of non-collinear vectors. In $\mathbb{R}^3$, a coordinate system is given by a triple of non-coplanar vectors. In a coordinate system, a vector decomposes along the basis vectors. The same will hold for a basis of a vector space.

Definition (Basis of a vector space). Let E be a K -vector space. A family $B = (v_1, v_2, \dots, v_n)$ of vectors in E is called a basis of E if B is both linearly independent and generating.



Theorem *Let $B = (v_1, v_2, \dots, v_n)$ be a basis of the vector space E . Every vector $v \in E$ can be expressed uniquely as a linear combination of elements of B . In other words, there exist unique scalars $\lambda_1, \dots, \lambda_n \in K$ such that:*

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n.$$

Continuation from Previous Notes

1. $(\lambda_1, \dots, \lambda_n)$ are called the *coordinates* of the vector v in the basis $\mathcal{B}$.
2. Note that for a basis $\mathcal{B} = (v_1, v_2, \dots, v_n)$, we introduce an ordering of the vectors. Of course, permuting the vectors still yields a basis, but the coordinates must be permuted accordingly.
3. Note that the mapping

$$\begin{aligned} K^n &\rightarrow E \\ (\lambda_1, \dots, \lambda_n) &\mapsto \lambda_1 v_1 + \dots + \lambda_n v_n \end{aligned}$$

is an isomorphism from the vector space K^n to the vector space E .

Proof of the theorem

By definition, $\mathcal{B}$ is a generating family of E , so for every $v \in E$ there exist $\lambda_1, \dots, \lambda_n \in K$ such that

$$v = \lambda_1 v_1 + \dots + \lambda_n v_n.$$

This proves existence.

It remains to show uniqueness. Suppose $\mu_1, \mu_2, \dots, \mu_n \in K$ are other scalars such that

$$v = \mu_1 v_1 + \dots + \mu_n v_n.$$

Then subtracting, we get:

$$(\lambda_1 - \mu_1)v_1 + (\lambda_2 - \mu_2)v_2 + \dots + (\lambda_n - \mu_n)v_n = 0.$$

Since $\mathcal{B} = \{v_1, \dots, v_n\}$ is linearly independent, this implies

$$\lambda_1 - \mu_1 = 0, \dots, \lambda_n - \mu_n = 0,$$

hence $\lambda_1 = \mu_1, \dots, \lambda_n = \mu_n$.

Example 1. Let

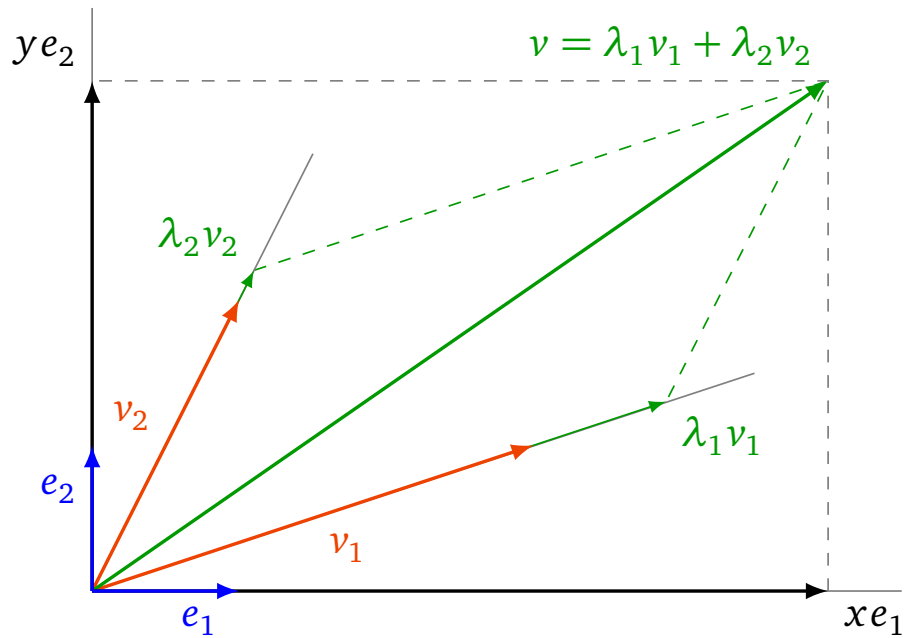
$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then (e_1, e_2) is a basis of $\mathbb{R}^2$, called the canonical basis of $\mathbb{R}^2$.

2. Let

$$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

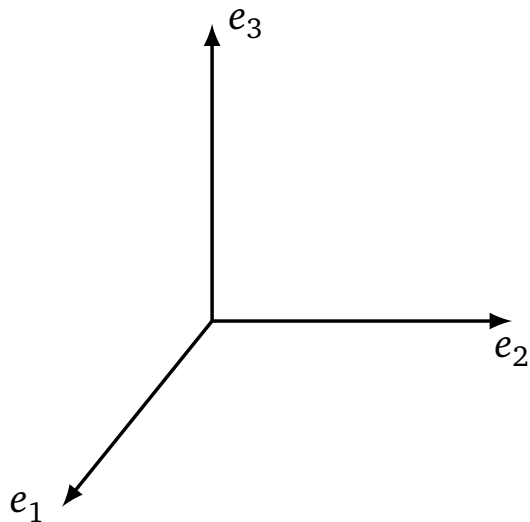
Then (v_1, v_2) also forms a basis of $\mathbb{R}^2$.



3. Similarly, in $\mathbb{R}^3$, if

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

then (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$.



Example *Let*

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

We show that the family $B = (v_1, v_2, v_3)$ is a basis of $\mathbb{R}^3$. We reduce the problem to solving linear systems.

1. Generating family: *Let*

$$v = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

be any vector in $\mathbb{R}^3$. We look for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3.$$

This gives:

$$\begin{cases} \lambda_1 + \lambda_2 = a_1 \\ \lambda_1 + \lambda_3 = a_2 \\ \lambda_2 + \lambda_3 = a_3 \end{cases}$$

We must show that this system has a solution for all a_1, a_2, a_3 .

2. Linear independence: *We show that the only solution to*

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$$

is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This is equivalent to solving the homogeneous system:

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_1 + \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{cases}$$

One can check that the only solution is the trivial one. Hence $\mathcal{B}$ is linearly independent. Since it is also generating, it is a basis.

2.6.1 Existence of a Basis

We now state an existence theorem for a finite basis. In what follows, vector spaces are assumed not to be reduced to $\{0\}$.

Theorem (Existence of a basis). *Every vector space that admits a finite generating family admits a basis.*