

Tutorial N° 1

Exercise 1. 1. Show that the vectors

$$v_1 = (0, 1, 1), \quad v_2 = (1, 0, 1), \quad v_3 = (1, 1, 0)$$

form a basis of $\mathbb{R}^3$. Find the components of the vector $w = (1, 1, 1)$ in this basis (v_1, v_2, v_3) .

2. Show that the vectors

$$v_1 = (1, 1, 1), \quad v_2 = (-1, 1, 0), \quad v_3 = (1, 0, -1)$$

form a basis of $\mathbb{R}^3$. Find the components of the vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ and $w = (1, 2, -3)$ in this basis (v_1, v_2, v_3) .

Exercise 2. 1. In $\mathbb{R}^3$, give an example of a linearly independent family that is not generating (spanning).

2. In $\mathbb{R}^3$, give an example of a generating family (spanning set) that is not linearly independent.

Exercise 3. Determine for which values of $t \in \mathbb{R}$ the vectors

$$\{(1, 0, t), (1, 1, t), (t, 0, 1)\}$$

form a basis of $\mathbb{R}^3$.

Exercise 4. 1. Show that the vectors

$$v_1 = (1, -1, i), \quad v_2 = (-1, i, 1), \quad v_3 = (i, 1, -1)$$

form a basis of $\mathbb{C}^3$.

2. Calculate the coordinates of

$$v = (1 + i, 1 - i, i)$$

in this basis.

Exercise 5. Consider, in $\mathbb{R}^4$, the vectors:

$$v_1 = (1, 2, 3, 4), \quad v_2 = (1, 1, 1, 3), \quad v_3 = (2, 1, 1, 1),$$

$$v_4 = (-1, 0, -1, 2), \quad v_5 = (2, 3, 0, 1)$$

Let F be the vector space generated by $\{v_1, v_2, v_3\}$ and let G be the one generated by $\{v_4, v_5\}$. Compute the respective dimensions of F, G

Solution to Exercise 1

1. To show that the family $\{v_1, v_2, v_3\}$ is a basis, we will show that this family is linearly independent and generating (spanning).
- (a) Let us show that the family $\{v_1, v_2, v_3\}$ is linearly independent. Consider a zero linear combination $av_1 + bv_2 + cv_3 = 0$; we must show that the coefficients a, b, c are zero. Here the zero vector is $\mathbf{0} = (0, 0, 0)$.

$$\begin{aligned} av_1 + bv_2 + cv_3 &= (0, 0, 0) \\ \iff a(0, 1, 1) + b(1, 0, 1) + c(1, 1, 0) &= (0, 0, 0) \\ \iff (b + c, a + c, a + b) &= (0, 0, 0) \end{aligned}$$

Thus the coefficients satisfy $a = b = c = 0$; this proves that the family is linearly independent.

- (b) Let us show that the family $\{v_1, v_2, v_3\}$ is generating (spanning). For any vector $v = (x, y, z) \in \mathbb{R}^3$, we must find $a, b, c \in \mathbb{R}$ such that $av_1 + bv_2 + cv_3 = v$.

$$\begin{aligned} av_1 + bv_2 + cv_3 &= v \\ \iff a(0, 1, 1) + b(1, 0, 1) + c(1, 1, 0) &= (x, y, z) \\ \iff (b + c, a + c, a + b) &= (x, y, z) \\ \iff \begin{cases} b + c = x \\ a + c = y \\ a + b = z \end{cases} \end{aligned}$$

Solving the system:

$$a = \frac{1}{2}(-x + y + z), \quad b = \frac{1}{2}(x - y + z), \quad c = \frac{1}{2}(x + y - z).$$

For these values, we have $av_1 + bv_2 + cv_3 = (x, y, z) = v$. Therefore the family $\{v_1, v_2, v_3\}$ is generating.

- (c) The family is both linearly independent and generating, so it is a basis.
- (d) To write $w = (1, 1, 1)$ in the basis (v_1, v_2, v_3) , we can solve the system corresponding to $av_1 + bv_2 + cv_3 = w$. But we have already solved it for any vector (x, y, z) ; in particular for the vector $(1, 1, 1)$, the solution is

$$a = \frac{1}{2}, \quad b = \frac{1}{2}, \quad c = \frac{1}{2}.$$

In other words,

$$\frac{1}{2}v_1 + \frac{1}{2}v_2 + \frac{1}{2}v_3 = w.$$

The coordinates of w in the basis (v_1, v_2, v_3) are therefore $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

2. To show that the family is linearly independent and generating, the calculations are similar to those in the previous question. Denote by D the basis (v_1, v_2, v_3) . Next, express e_1 in this basis. The calculations give:

$$e_1 = \frac{1}{3}v_1 - \frac{1}{3}v_2 + \frac{1}{3}v_3.$$

Its coordinates in the basis D are $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$.

$$e_2 = \frac{1}{3}v_1 + \frac{2}{3}v_2 + \frac{1}{3}v_3.$$

Its coordinates in D are $(\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$.

$$e_3 = \frac{1}{3}v_1 - \frac{1}{3}v_2 - \frac{2}{3}v_3.$$

Its coordinates in D are $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$.

Notice that $w = (1, 2, -3)$ is actually $w = e_1 + 2e_2 - 3e_3$. Therefore, from our previous calculations:

$$\begin{aligned} w &= \frac{1}{3}v_1 - \frac{1}{3}v_2 + \frac{1}{3}v_3 \\ &\quad + 2\left(\frac{1}{3}v_1 + \frac{2}{3}v_2 + \frac{1}{3}v_3\right) \\ &\quad - 3\left(\frac{1}{3}v_1 - \frac{1}{3}v_2 - \frac{2}{3}v_3\right) \\ &= 2v_2 + 3v_3. \end{aligned}$$

The coordinates of w in D are $(0, 2, 3)$.

Solution to Exercise 2

1. For example, the family $\{(1, 0, 0), (0, 1, 0)\}$ is linearly independent in $\mathbb{R}^3$ but not generating.
2. The family $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$ is generating in $\mathbb{R}^3$ but not linearly independent.

Solution to Exercise 3

When the number of vectors equals the dimension of the space, we have equivalences between being a linearly independent family, being a generating family, and hence being a basis.

Three vectors in $\mathbb{R}^3$ therefore form a basis if and only if they form a linearly independent family. Let us check when this is the case.

$$\begin{aligned}
a(1, 0, t) + b(1, 1, t) + c(t, 0, 1) &= (0, 0, 0) \\
\iff (a + b + tc, b, at + bt + c) &= (0, 0, 0) \\
\iff \begin{cases} a + b + tc = 0 \\ b = 0 \\ at + bt + c = 0 \end{cases}
\end{aligned}$$

From $b = 0$, we get:

$$\begin{cases} a + tc = 0 \\ at + c = 0 \end{cases}$$

Substituting $a = -tc$ into $at + c = 0$:

$$(-tc)t + c = 0 \iff c(1 - t^2) = 0.$$

First case: if $t \neq \pm 1$. Then $t^2 - 1 \neq 0$ and the only solution to the system is $(a = 0, b = 0, c = 0)$. In this case, the family is linearly independent and therefore also a basis.

Second case: if $t = \pm 1$. Then the last equation disappears, and there exist nontrivial solutions (for example, if $t = 1$, $(a = 1, b = 0, c = -1)$ is a solution). The family is not linearly independent and therefore not a basis.

Thus the vectors form a basis if and only if $t \in \mathbb{R} \setminus \{-1, 1\}$.

Solution to Exercise 4

1. This is indeed a basis. Since we have three vectors and we wish to show that they form a basis of a vector space of dimension 3, it suffices to show either that the family is linearly independent or that it is generating (these conditions are equivalent for n vectors in an n -dimensional vector space). It is simpler to show that the family is linearly independent. Consider a zero linear combination $av_1 + bv_2 + cv_3 = 0$; we must show that $a = b = c = 0$. But here the base field is $K = \mathbb{C}$, so a, b, c are complex numbers.

$$a(1, -1, i) + b(-1, i, 1) + c(i, 1, -1) = (0, 0, 0)$$

This gives the system:

$$\begin{cases} a - b + ic = 0 \\ -a + ib + c = 0 \\ ia + b - c = 0 \end{cases}$$

Solving yields $a = 0, b = 0, c = 0$. Therefore the family (v_1, v_2, v_3) is linearly independent, hence also generating; it is therefore a basis of $\mathbb{C}^3$.

2. We look for $a, b, c \in \mathbb{C}$ such that

$$av_1 + bv_2 + cv_3 = v = (1 + i, 1 - i, i).$$

Solving the system gives:

$$a = 0, \quad b = \frac{1}{2}(1 - i), \quad c = \frac{1}{2}(1 - 3i).$$

Thus

$$v = \frac{1}{2}(1-i)v_2 + \frac{1}{2}(1-3i)v_3.$$

Consequently, the coordinates of v in the basis (v_1, v_2, v_3) are

$$\left(0, \frac{1}{2}(1-i), \frac{1}{2}(1-3i)\right).$$

Solution to Exercise 5

1. G is generated by two vectors, so $\dim G \leq 2$. Clearly v_4 and v_5 are not linearly dependent, so $\dim G \geq 2$; hence $\dim G = 2$.
2. F is generated by three vectors, so $\dim F \leq 3$. A calculation shows that the family $\{v_1, v_2, v_3\}$ is linearly independent, hence $\dim F \geq 3$ and therefore $\dim F = 3$.