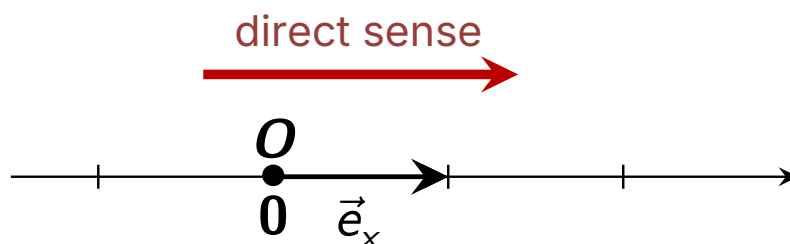


2.7 Orientation

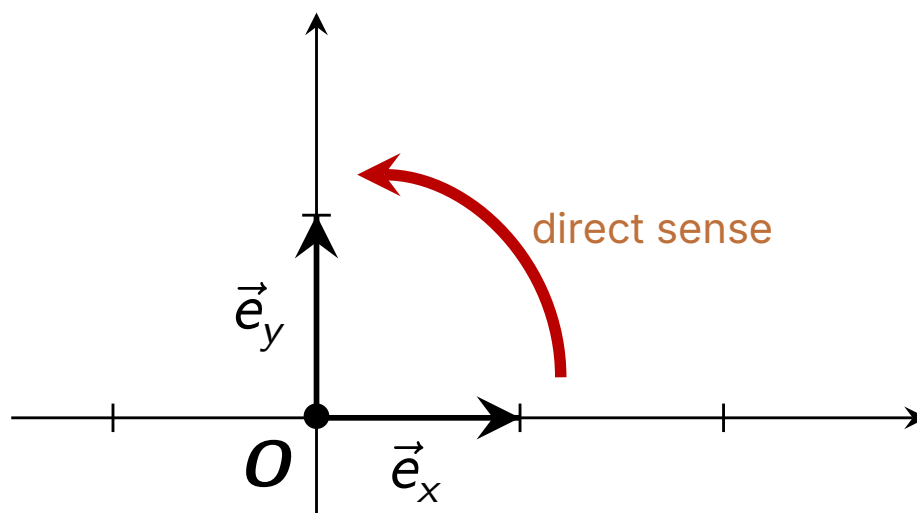
2.7.1 Orientation of a Line

To orient a line, we choose an origin O and a direction of travel on the line (2 possible orientations). The chosen unit vector $\vec{e}_x$ gives the orientation. This defines an oriented normalized coordinate system $(O; \vec{e}_x)$.



2.7.2 Orientation of a Plane

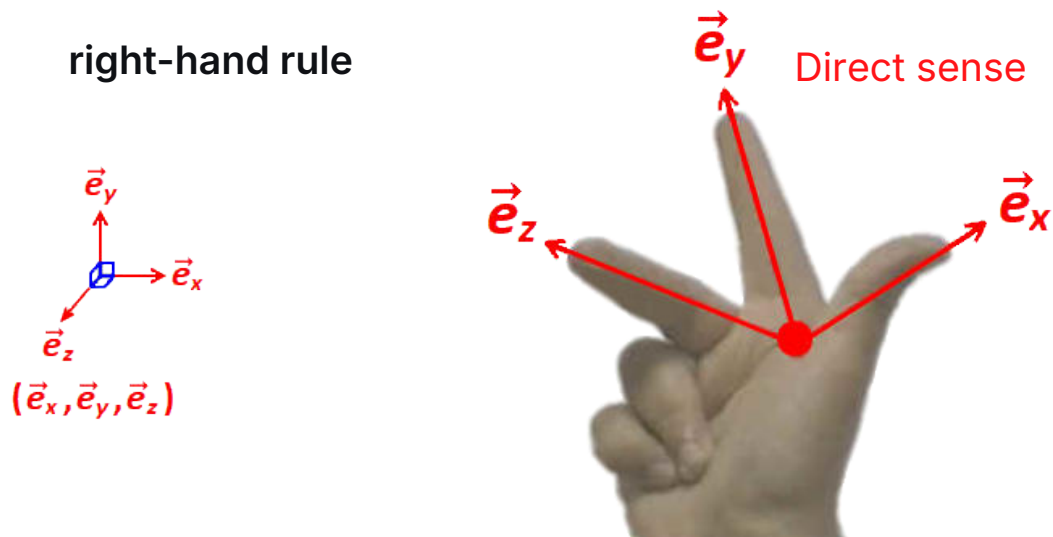
We choose a normalized axis $(O; \vec{e}_x)$, then a second axis through O with normalized coordinate system $(O; \vec{e}_y)$, perpendicular to the first. We choose a direction of rotation to go from $\vec{e}_x$ to $\vec{e}_y$; this is the direct or trigonometric direction. This yields a direct orthonormal coordinate system $(O; \vec{e}_x, \vec{e}_y)$.



2.7.3 Orientation of Space

Once a plane in space is equipped with an orthonormal coordinate system $(O; \vec{e}_x, \vec{e}_y)$, there are two opposite choices for the last unit vector $\vec{e}_z$ orthogonal to the first two, to orient space. In physics, the direct sense of the coordinate system $(O; \vec{e}_x, \vec{e}_y, \vec{e}_z)$ corresponds to the right-hand rule:

- thumb: $\vec{e}_x$
- index finger: $\vec{e}_y$
- middle finger: $\vec{e}_z$

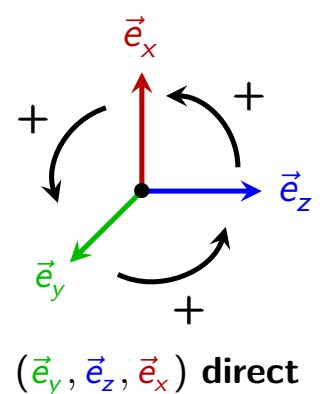
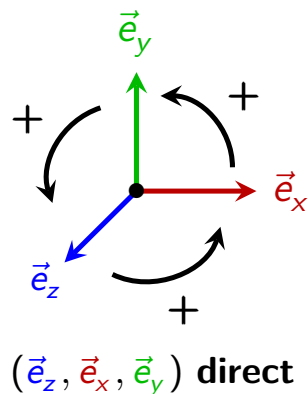
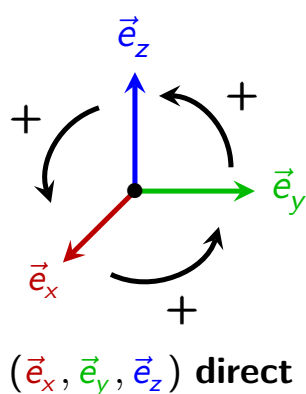


We obtain a coordinate system $(O; \vec{e}_x, \vec{e}_y, \vec{e}_z)$ in which the three vectors are unit and pairwise orthogonal, oriented according to the above rule.
a **direct orthonormal coordinate system**.

2.7.4 Orientation and Permutations

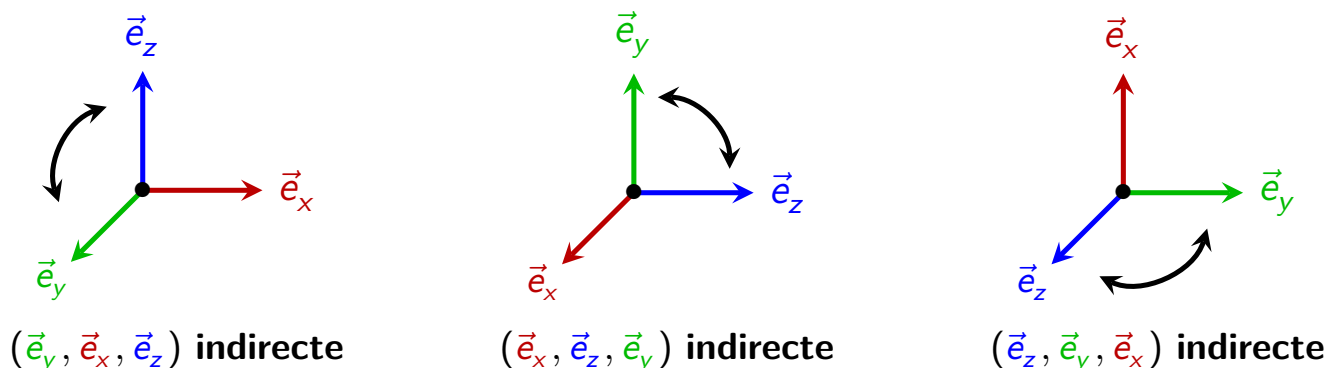
- A cyclic permutation preserves orientation:

$$(\vec{e}_x, \vec{e}_y, \vec{e}_z) \rightarrow (\vec{e}_y, \vec{e}_z, \vec{e}_x) \rightarrow (\vec{e}_z, \vec{e}_x, \vec{e}_y).$$



- A transposition (swapping two vectors) reverses orientation:

$$(\vec{e}_x, \vec{e}_y, \vec{e}_z) \rightarrow (\vec{e}_y, \vec{e}_x, \vec{e}_z).$$



3 Dimension of a Vector Space

Definition A K -vector space E that admits a basis with a finite number of elements is said to be finite-dimensional.

By the Existence Theorem (Theorem 3), this is equivalent to the existence of a finite generating family. We can speak of the dimension of a vector space thanks to the following theorem:

Theorem (Dimension theorem). All bases of a finite-dimensional vector space E have the same number of elements.

We will detail the proof a bit later.

Definition The dimension of a finite-dimensional vector space E , denoted $\dim E$, is by definition the number of elements in any basis of E .

Methodology. To determine the dimension of a vector space, it suffices to find one basis of E (a family that is both linearly independent and generating): the cardinality (number of elements) of this family gives the dimension of E . The dimension theorem proves that even if we chose a different basis, both bases would have the same number of elements.

Convention. We assign dimension 0 to the zero vector space $\{0\}$.

Example 1. The canonical basis of $\mathbb{R}^2$ is

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

Hence the dimension of $\mathbb{R}^2$ is 2.

2. The vectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

also form a basis of $\mathbb{R}^2$, illustrating that another basis contains the same number of elements.

3. More generally, K^n has dimension n , because its canonical basis $(e_1, e_2, \dots, e_n)$ contains n elements.

4. $\dim \mathbb{R}_n[X] = n + 1$, because a basis of $\mathbb{R}_n[X]$ is $(1, X, X^2, \dots, X^n)$, which contains $n + 1$ elements.

3.1 Vector Geometry of Space

3.1.1 Vectors

A vector in the plane or space is characterized by its direction, sense (orientation), and length (or norm).

A basis of the plane is given by two non-collinear vectors.

A basis of space is given by three non-coplanar vectors.

It allows any vector in the plane or space to be located using its components (sometimes also called coordinates).

Components/bases in space: various notations

- $\vec{u} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the basis $(\vec{i}, \vec{j}, \vec{k})$ means: $\vec{u} = x\vec{i} + y\vec{j} + z\vec{k}$ — simple notation in dimension 2 or 3.

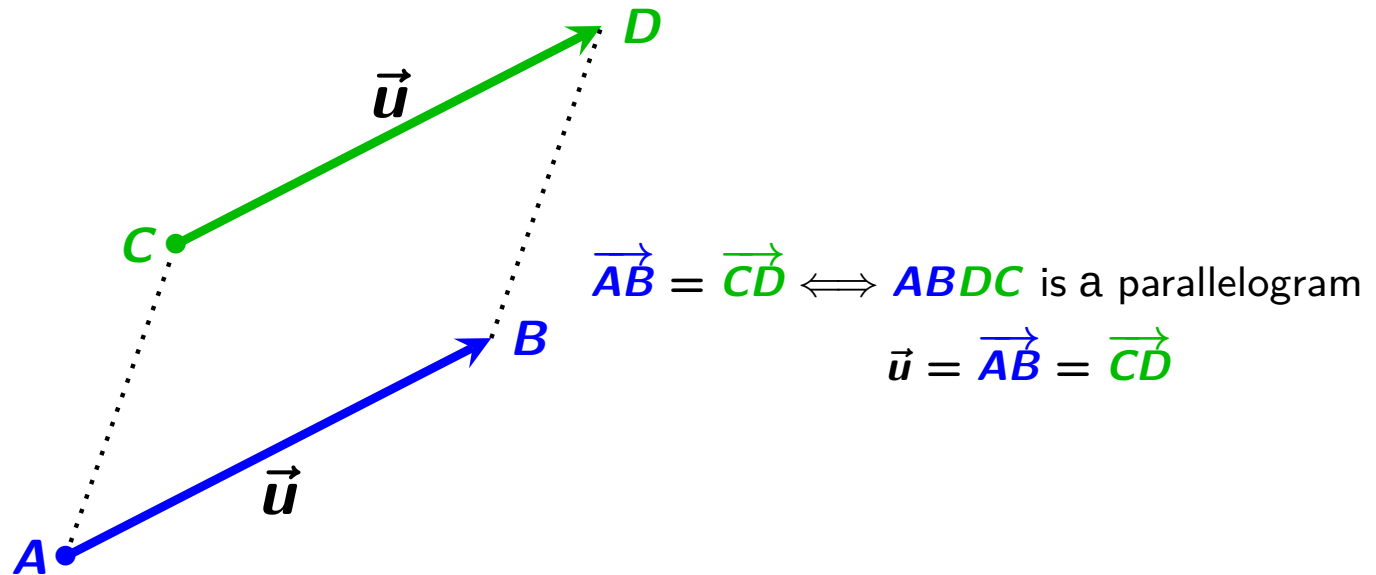
- $\vec{u} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in the basis (e_1, e_2, e_3) means: $\vec{u} = x_1e_1 + x_2e_2 + x_3e_3$ — generalizable to higher dimensions.

- $\vec{u} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ in the basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ means: $\vec{u} = u_x\vec{e}_x + u_y\vec{e}_y + u_z\vec{e}_z$ — useful for changes of coordinate systems (Cartesian, polar, cylindrical, spherical, etc.).

3.1.2 Vectors and Points

A vector in the plane or space is geometrically represented by an ordered pair of points (A, B) with an arrow indicating direction: $\vec{u} = \overrightarrow{AB}$.

It is thus represented by a directed line segment. Two parallel directed line segments of the same length and same direction represent the same vector (parallelogram rule).



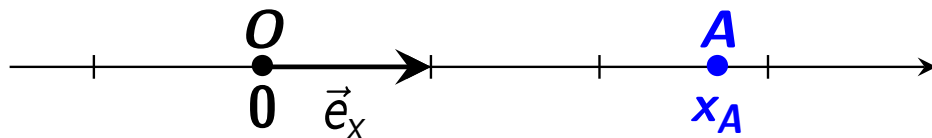
3.2 Coordinates / Coordinate Systems

A coordinate system (frame) in space is given by a point O and a basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$; it is written $(O; \vec{e}_x, \vec{e}_y, \vec{e}_z)$. It allows any point in space to be located using its coordinates.

3.2.1 Positioning on a Line

In a coordinate system $(O; \vec{e}_x)$, any point A on the line is located by its abscissa x_A :

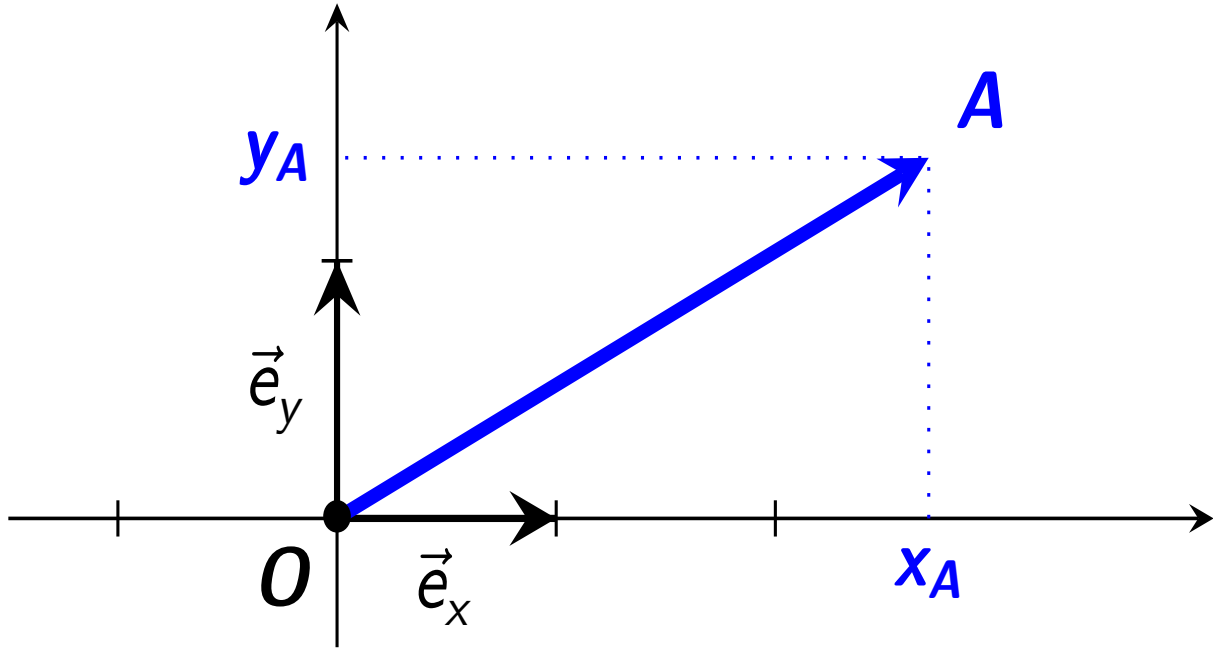
$$\overrightarrow{OA} = x_A \vec{e}_x.$$



3.2.2 Positioning in a Plane

In a coordinate system $(O; \vec{e}_x, \vec{e}_y)$, any point A in the plane is located by its abscissa x_A and ordinate y_A :

$$\overrightarrow{OA} = x_A \vec{e}_x + y_A \vec{e}_y.$$



3.2.3 Positioning in Space

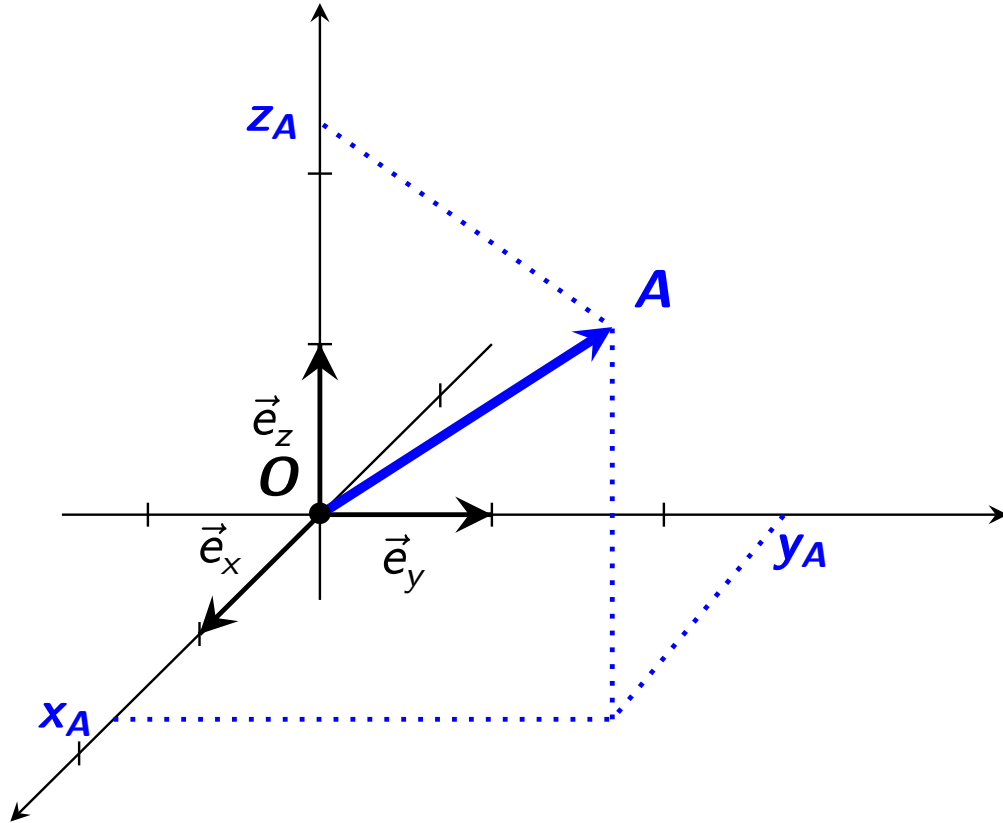
In a coordinate system $(O; \vec{e}_x, \vec{e}_y, \vec{e}_z)$, any point A in space is located by its abscissa x_A , ordinate y_A , and height (z-coordinate) z_A :

$$\overrightarrow{OA} = x_A \vec{e}_x + y_A \vec{e}_y + z_A \vec{e}_z.$$

We usually write the coordinates of a point in a row and the components of a vector in a column:

$$A(x_A, y_A, z_A), \quad \overrightarrow{OA} \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix}.$$

$$O(0,0,0), \quad \vec{e}_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



3.2.4 Vectors and Points

If (x_A, y_A, z_A) and (x_B, y_B, z_B) are the coordinates of A and B in the coordinate system $(O; \vec{e}_x, \vec{e}_y, \vec{e}_z)$, we write $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$.

We have:

$$\overrightarrow{OA} = x_A \vec{e}_x + y_A \vec{e}_y + z_A \vec{e}_z, \quad \overrightarrow{OB} = x_B \vec{e}_x + y_B \vec{e}_y + z_B \vec{e}_z.$$

Then the vector $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ has components in the basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$:

$$\overrightarrow{AB} \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}.$$

Note: Sometimes we write components in a row:

$$\overrightarrow{AB}(x_B - x_A, y_B - y_A, z_B - z_A).$$

We also sometimes write $\overrightarrow{AB} = B - A$, consistent with the relation between coordinates of A and B and components of $\overrightarrow{AB}$ described above.

4 Operations on Vectors and Projection of Vectors onto Axes in a Cartesian Coordinate System

4.1 Operations on Vectors

We generalize these notions by considering spaces of dimension n for any positive integer $n = 1, 2, 3, 4, \dots$. The elements of an n -dimensional space are n -tuples

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

of real numbers. The space of dimension n is denoted $\mathbb{R}^n$. As in dimensions 2 and 3, the n -tuple denotes both a point and a vector in the n -dimensional space.

Let

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

be two vectors in $\mathbb{R}^n$.

Definition 1. *Sum of two vectors.* Their sum is by definition the vector

$$u + v = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}.$$

2. *Scalar multiplication.* Let $\lambda \in \mathbb{R}$ (called a scalar):

$$\lambda u = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \vdots \\ \lambda u_n \end{pmatrix}.$$

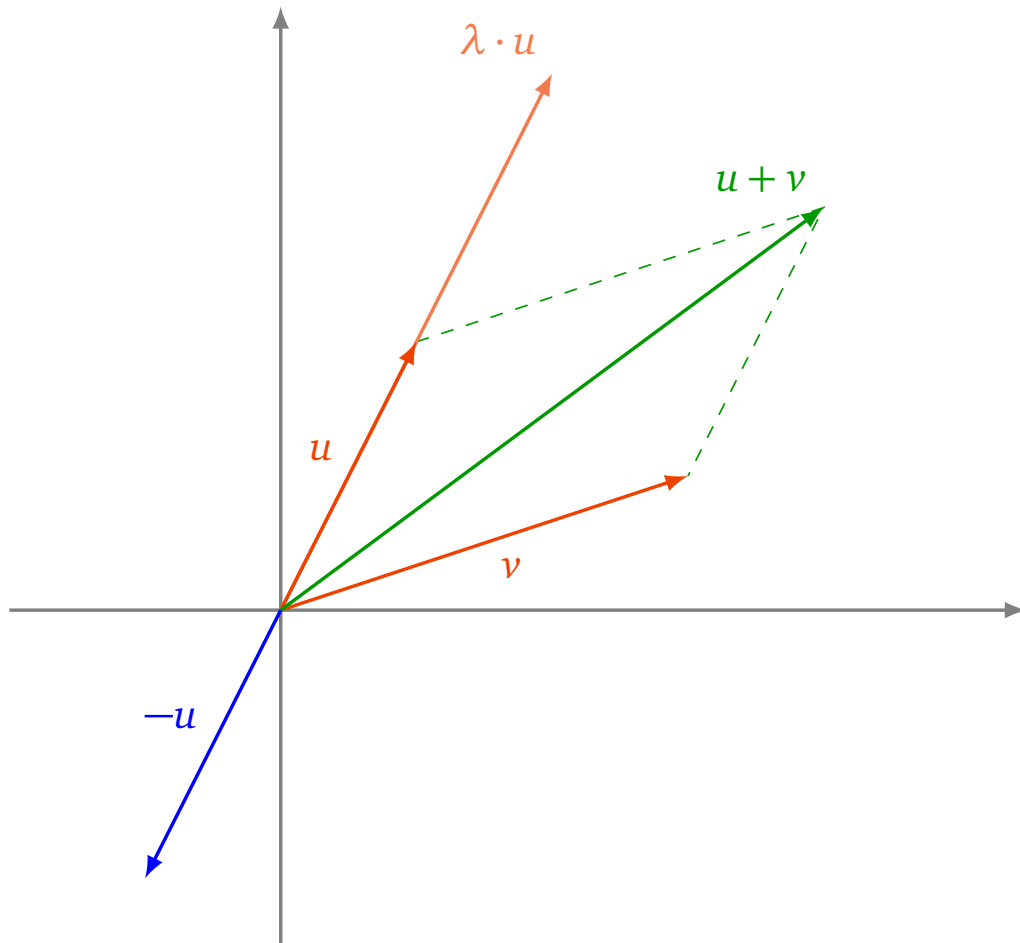
3. The **zero vector** of $\mathbb{R}^n$ is

$$0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

4. The **opposite** of vector u is

$$-u = \begin{pmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{pmatrix}.$$

Initially, you may write $\vec{u}, \vec{v}, \vec{0}$ instead of $u, v, 0$. But you should quickly get used to the notation without arrows. Similarly, if λ is a scalar and u a vector, we will often write λu instead of $\lambda \cdot u$.



4.2 Properties of Vector Spaces

Theorem *Let*

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

be vectors in $\mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$. Then:

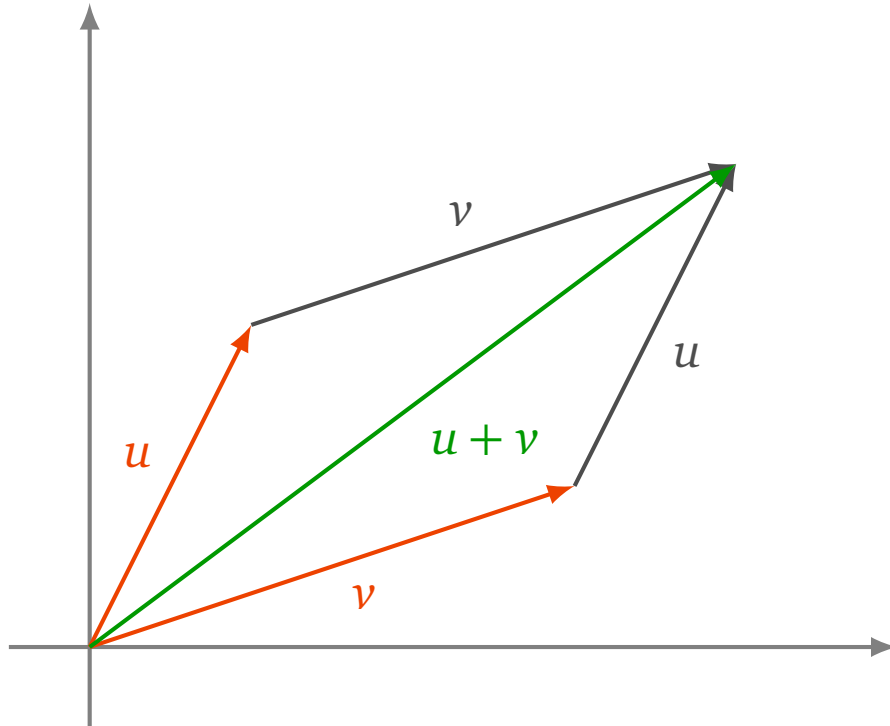
1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. $u + 0 = 0 + u = u$
4. $u + (-u) = 0$

5. $1 \cdot u = u$

6. $\lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u$

7. $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$

8. $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$



Each of these properties follows directly from the definition of vector addition and scalar multiplication. These eight properties make $\mathbb{R}^n$ a vector space. In the general framework, these eight properties define what a vector space is.

4.3 Dot Product (Scalar Product)

4.3.1 Definition (Dot Product / Norm)

1. Let $\vec{u}$ and $\vec{v}$ be two vectors in space. Their dot product is the real number:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \times \|\vec{v}\| \times \cos(\vec{u}, \vec{v})$$

where $(\vec{u}, \vec{v})$ is the angle between $\vec{u}$ and $\vec{v}$.

Note: The angle is not necessarily oriented.

2. The norm is derived from the squared dot product:

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}.$$

A vector is called *normalized* or *unit* when its norm equals 1.

4.3.2 Property (Bilinearity)

1. The dot product is symmetric: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
2. The dot product is bilinear:

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}, \quad (\lambda \vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{v}) \quad (\lambda \in \mathbb{R}).$$

3. The dot product is positive definite:

$$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2 > 0 \quad \text{for } \vec{u} \neq \vec{0}.$$

4.3.3 Property (Analytic Expression)

Let an orthonormal basis of space be given, and let $\vec{u}$ and $\vec{v}$ have components in this basis. Then the dot product and norm are expressed as:

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z, \quad \|\vec{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2}.$$

Remark (Unit Vector). For any nonzero vector $\vec{u}$, the vector $\frac{1}{\|\vec{u}\|}\vec{u}$ is a unit vector with the same direction and sense as $\vec{u}$.

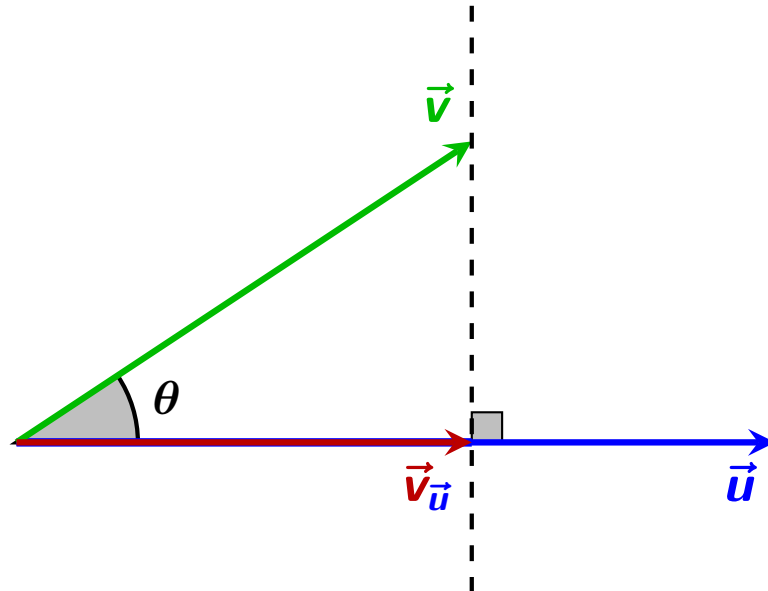
4.3.4 Property (Orthogonal Projection of a Vector onto Another)

The orthogonal projection $\vec{v}_{\parallel}$ of a vector $\vec{v}$ onto a nonzero vector $\vec{u}$ is given by:

$$\vec{v}_{\parallel} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}.$$

If we denote by $v_{\parallel}$ the algebraic measure of $\vec{v}_{\parallel}$ on the axis oriented by $\vec{u}$, then we have:

$$v_{\parallel} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}.$$



Consequently, $v_{\parallel}$ and $\vec{u} \cdot \vec{v}$ have the same sign.

Special case: when $\vec{u}$ is a unit vector, the formula simplifies to

$$\vec{v}_{\parallel} = (\vec{u} \cdot \vec{v})\vec{u}.$$

Proof of the projection

By trigonometry in a right triangle,

$$v_{\parallel} = \|\vec{v}\| \times \cos \theta$$

where $\theta = (\vec{u}, \vec{v})$ is the geometric angle between $\vec{u}$ and $\vec{v}$.

Moreover, we have

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \times \|\vec{v}\|},$$

therefore

$$v_{\parallel} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}.$$

We then obtain the vector $\vec{v}_{\parallel}$ along the line directed by the unit vector $\frac{1}{\|\vec{u}\|}\vec{u}$ as

$$\vec{v}_{\parallel} = v_{\parallel} \left(\frac{1}{\|\vec{u}\|} \vec{u} \right) = \frac{v_{\parallel}}{\|\vec{u}\|} \vec{u},$$

hence

$$\vec{v}_{\parallel} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}.$$

Example (Projections onto an orthonormal basis). *Let $\vec{u}$ be a vector with components*

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

in the orthonormal basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ of space, i.e.,

$$\vec{u} = u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z.$$

We can obtain its components using dot products (projections onto the basis vectors) as follows:

$$u_x = \vec{u} \cdot \vec{e}_x, \quad u_y = \vec{u} \cdot \vec{e}_y, \quad u_z = \vec{u} \cdot \vec{e}_z.$$

4.3.5 Geometric Applications

Testing orthogonality of two vectors:

$$\vec{u} \text{ and } \vec{v} \text{ are orthogonal} \iff \vec{u} \cdot \vec{v} = 0.$$

Testing collinearity of two vectors:

$$\vec{u} \text{ and } \vec{v} \text{ are collinear} \iff \vec{u} \cdot \vec{v} = \pm \|\vec{u}\| \cdot \|\vec{v}\|.$$

The vectors have the same sense if the dot product is positive.

Example (Plane orthogonal to a vector). *We work in an orthonormal basis of space. Let a, b, c be three real numbers not all zero, and let*

$$\vec{u} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

be a fixed vector. Then the set of vectors

$$\vec{v} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

orthogonal to $\vec{u}$ is characterized by the relation:

$$ax + by + cz = 0.$$

This is a Cartesian equation of the vector plane orthogonal to $\vec{u}$.

4.4 Cross Product (Vector Product)

4.4.1 Definition (Cross Product)

Let $\vec{u}$ and $\vec{v}$ be two vectors in oriented space. Their cross product is the vector $\vec{u} \wedge \vec{v}$ defined as follows:

If $\vec{u}$ and $\vec{v}$ are collinear or if either is zero, then

$$\vec{u} \wedge \vec{v} = \vec{0}.$$

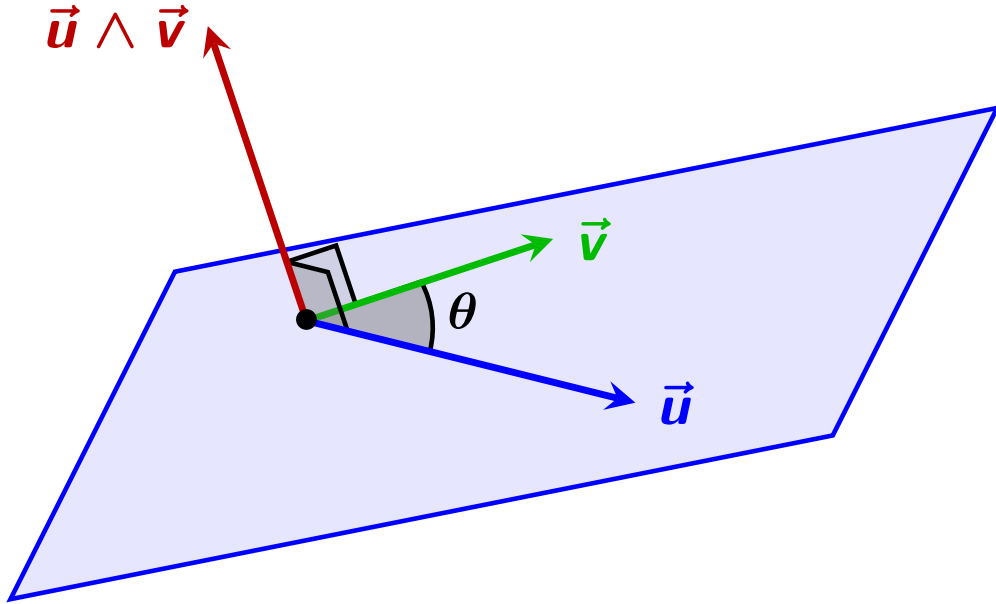
If $\vec{u}$ and $\vec{v}$ are neither zero nor collinear, then $\vec{u} \wedge \vec{v}$ is the unique vector with the following characteristics:

1. **Direction:** $\vec{u} \wedge \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
2. **Sense:** the basis $(\vec{u}, \vec{v}, \vec{u} \wedge \vec{v})$ is direct.

3. **Length:**

$$\|\vec{u} \wedge \vec{v}\| = \|\vec{u}\| \times \|\vec{v}\| \times |\sin(\vec{u}, \vec{v})|.$$

Remark 3.1. *The cross product of two vectors exists only in dimension 3!*



4.4.2 Property (Bilinearity)

1. The cross product is antisymmetric:

$$\vec{u} \wedge \vec{v} = -\vec{v} \wedge \vec{u}.$$

2. The cross product is bilinear, i.e., linear in each variable:

$$(\vec{u} + \vec{v}) \wedge \vec{w} = \vec{u} \wedge \vec{w} + \vec{v} \wedge \vec{w}, \quad (\lambda \vec{u}) \wedge \vec{v} = \lambda(\vec{u} \wedge \vec{v}).$$

3. In a direct orthonormal basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, let

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}.$$

Then the cross product $\vec{u} \wedge \vec{v}$ has components:

$$\vec{u} \wedge \vec{v} = \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}.$$

In particular, if $\vec{u}$ and $\vec{v}$ lie in the plane spanned by $(\vec{e}_x, \vec{e}_y)$, i.e., they have components

$$\vec{u} \begin{pmatrix} u_x \\ u_y \\ 0 \end{pmatrix}, \quad \vec{v} \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix},$$

then

$$\vec{u} \wedge \vec{v} = (u_x v_y - u_y v_x) \vec{e}_z.$$

4.4.3 Components: Mnemonic Procedure

Remembering the component expressions of a cross product can be difficult. A practical method is as follows:

1. Write the components of the vectors as column matrices.
2. Copy the first two components of each column below.
3. Note the positions of cross products and compute the differences:

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \wedge \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \rightarrow \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}$$

$\begin{matrix} u_x & v_x \\ u_y & v_y \end{matrix} \rightarrow u_y v_x - u_x v_y$
 $\begin{matrix} u_x & v_x \\ u_z & v_z \end{matrix} \rightarrow u_z v_x - u_x v_z$
 $\begin{matrix} u_y & v_y \\ u_z & v_z \end{matrix} \rightarrow u_x v_y - u_y v_x$

Then:

$$(\vec{u} \wedge \vec{v})_x = u_y v_z - u_z v_y, \quad (\vec{u} \wedge \vec{v})_y = u_z v_x - u_x v_z, \quad (\vec{u} \wedge \vec{v})_z = u_x v_y - u_y v_x.$$

We can also compute cross product components using a determinant. Let

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (2 \times 2 \text{ determinant}).$$

Then

$$\vec{u} \wedge \vec{v} = \begin{vmatrix} u_x & v_x & \vec{e}_x \\ u_y & v_y & \vec{e}_y \\ u_z & v_z & \vec{e}_z \end{vmatrix} = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix} \vec{e}_x - \begin{vmatrix} u_x & v_x \\ u_z & v_z \end{vmatrix} \vec{e}_y + \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} \vec{e}_z$$

$$\vec{u} \wedge \vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = (u_y v_z - u_z v_y) \vec{e}_x + (u_z v_x - u_x v_z) \vec{e}_y + (u_x v_y - u_y v_x) \vec{e}_z.$$

4.4.4 Derivation of the Analytic Expression

$$\begin{aligned}\vec{u} \wedge \vec{v} &= (u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z) \wedge (v_x \vec{e}_x + v_y \vec{e}_y + v_z \vec{e}_z) \\ &= u_x v_x (\vec{e}_x \wedge \vec{e}_x) + u_x v_y (\vec{e}_x \wedge \vec{e}_y) + u_x v_z (\vec{e}_x \wedge \vec{e}_z) \\ &\quad + u_y v_x (\vec{e}_y \wedge \vec{e}_x) + u_y v_y (\vec{e}_y \wedge \vec{e}_y) + u_y v_z (\vec{e}_y \wedge \vec{e}_z) \\ &\quad + u_z v_x (\vec{e}_z \wedge \vec{e}_x) + u_z v_y (\vec{e}_z \wedge \vec{e}_y) + u_z v_z (\vec{e}_z \wedge \vec{e}_z) \\ &= (u_y v_z - u_z v_y) \vec{e}_x + (u_z v_x - u_x v_z) \vec{e}_y + (u_x v_y - u_y v_x) \vec{e}_z.\end{aligned}$$

Remark In any direct orthonormal basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ of space, we have:

$$\vec{e}_x \wedge \vec{e}_y = \vec{e}_z, \quad \vec{e}_y \wedge \vec{e}_z = \vec{e}_x, \quad \vec{e}_z \wedge \vec{e}_x = \vec{e}_y,$$

and

$$\vec{e}_x \wedge \vec{e}_x = \vec{e}_y \wedge \vec{e}_y = \vec{e}_z \wedge \vec{e}_z = \vec{0}.$$

4.4.5 Geometric Applications

Testing collinearity of two vectors:

$$\vec{u} \text{ and } \vec{v} \text{ are collinear} \iff \vec{u} \wedge \vec{v} = \vec{0}.$$

Testing orthogonality of two vectors:

$$\vec{u} \text{ and } \vec{v} \text{ are orthogonal} \iff \|\vec{u} \wedge \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\|.$$

Testing Coplanarity of Three Vectors

$$\vec{u}, \vec{v}, \vec{w} \text{ are coplanar} \iff \vec{u} \cdot (\vec{v} \wedge \vec{w}) = 0.$$

(Equivalently, $\vec{v} \cdot (\vec{w} \wedge \vec{u}) = 0$ or $\vec{w} \cdot (\vec{u} \wedge \vec{v}) = 0$.)

Remark The quantity $\vec{u} \cdot (\vec{v} \wedge \vec{w})$ is called the scalar triple product of $\vec{u}, \vec{v}, \vec{w}$, sometimes denoted $[(\vec{u}, \vec{v}, \vec{w})]$. These three expressions are equal.

Computing a Vector Orthogonal to Two Non-Collinear Vectors

$$\vec{u} \wedge \vec{v}$$

Computing a Normal Vector to a Plane Defined by Three Points

$$A, B, C : \overrightarrow{AB} \wedge \overrightarrow{AC}$$