

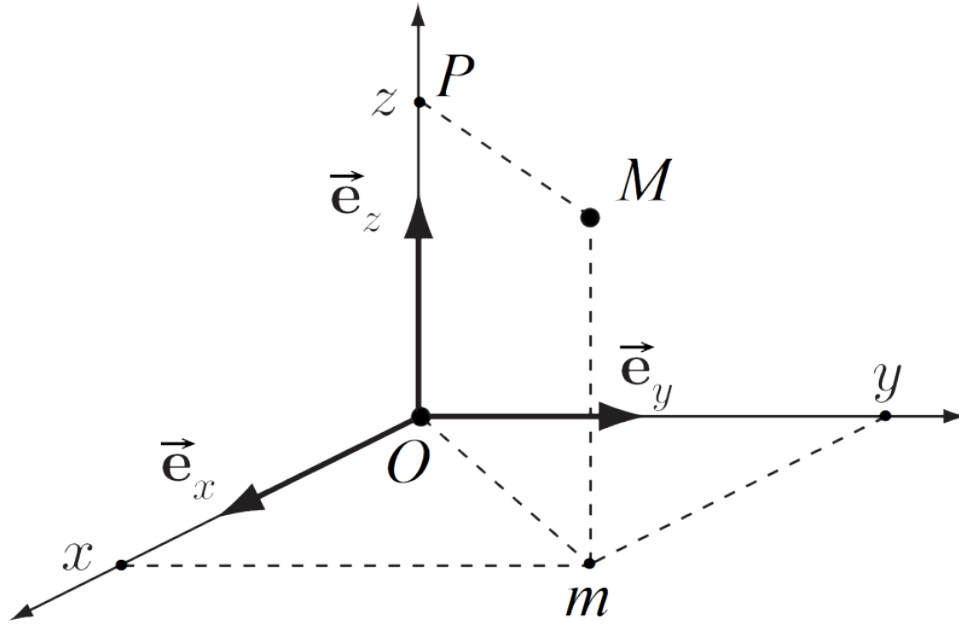
5 Vectors in Polar, Cylindrical, and Spherical Coordinates

5.1 Cartesian Coordinates

Throughout this section, we work in the plane or space equipped with a fixed, direct orthonormal coordinate system $(O; \vec{e}_x, \vec{e}_y)$ or $(O; \vec{e}_x, \vec{e}_y, \vec{e}_z)$.

Cartesian Coordinates In space (or the plane), any point M can be located by its Cartesian coordinates:

$$M(x, y, z) \quad \text{or} \quad M(x, y) \text{ in the plane.}$$



$$\overrightarrow{OM} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$

The points $m(x, y, 0)$ and $P(0, 0, z)$ are the orthogonal projections of M onto the plane (Oxy) and the axis (Oz) , respectively.

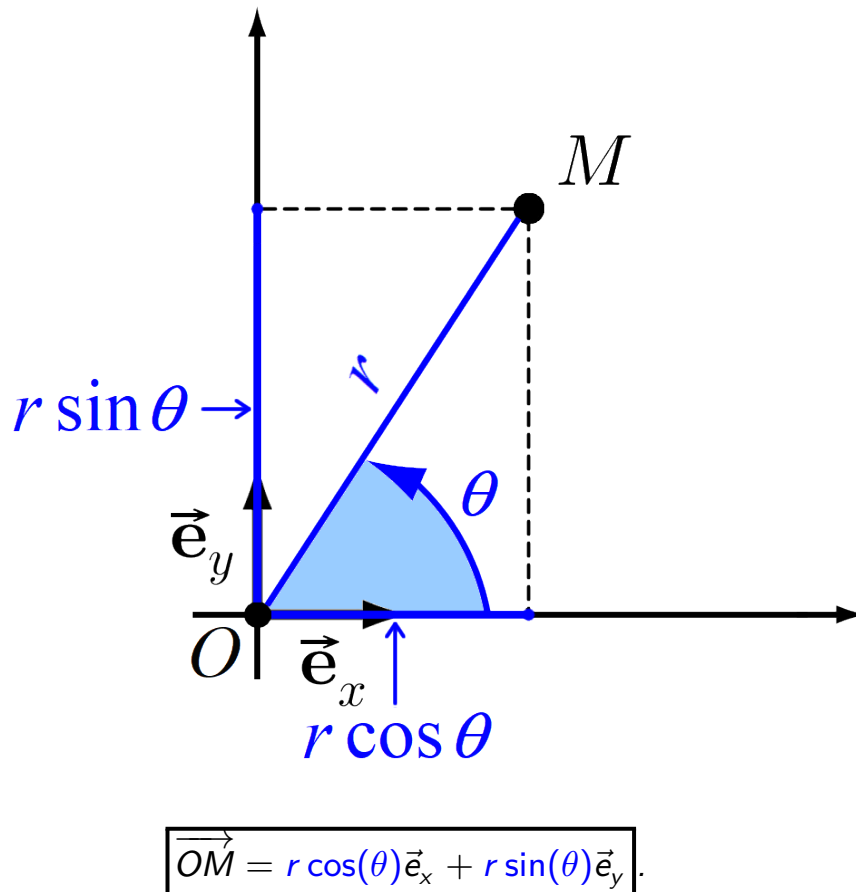
5.2 Polar Coordinates

Definition (Polar Coordinates). A point M in the plane can be located by its distance $r \geq 0$ from the origin O and, when $M \neq O$, its angle $\theta = (\vec{e}_x, \overrightarrow{OM})$ with $\theta \in [0, 2\pi[$.

The pair (r, θ) constitutes the polar coordinates of the point M .

With these notations, we have:

$$\overrightarrow{OM} = r \cos \theta \vec{e}_x + r \sin \theta \vec{e}_y.$$



The origin O and the axis $(O; \vec{e}_x)$ are called the *pole* and the *polar axis*, respectively. The point O has no unique polar coordinates.

Property (Conversion between Cartesian and Polar Coordinates). *To convert from Cartesian coordinates to polar coordinates and vice versa:*

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \iff \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases} \quad (x \neq 0)$$

(with appropriate quadrant adjustments).

Property (Coordinate Curves). *The coordinate curves in polar coordinates are obtained by fixing one of the coordinates:*

- The equation $r = \text{constant}$ gives a circle centered at O .
- The equation $\theta = \text{constant}$ gives a ray (half-line) starting at O .

More precisely, for fixed $r_0 > 0$ and $\theta_0 \in [0, 2\pi]$:

- The set of points $M(r_0, \theta)$ is the circle centered at O with radius r_0 .
- The set of points $M(r, \theta_0)$ is the ray from O with polar angle θ_0 .

5.3 Cylindrical Coordinates

Cylindrical coordinates in space are “polar coordinates plus altitude.”

Definition (Cylindrical Coordinates). We project the point M in space onto the plane (Oxy) at m and onto the axis (Oz) at P :

$$\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{OP},$$

and we locate the projection m by its polar coordinates in the plane:

$$\overrightarrow{OM} = r \cos(\theta) \vec{e}_x + r \sin(\theta) \vec{e}_y + z \vec{e}_z$$

with $r \in [0, +\infty[$, $\theta \in [0, 2\pi[$, $z \in \mathbb{R}$.

The triple (r, θ, z) constitutes the cylindrical coordinates of the point M .

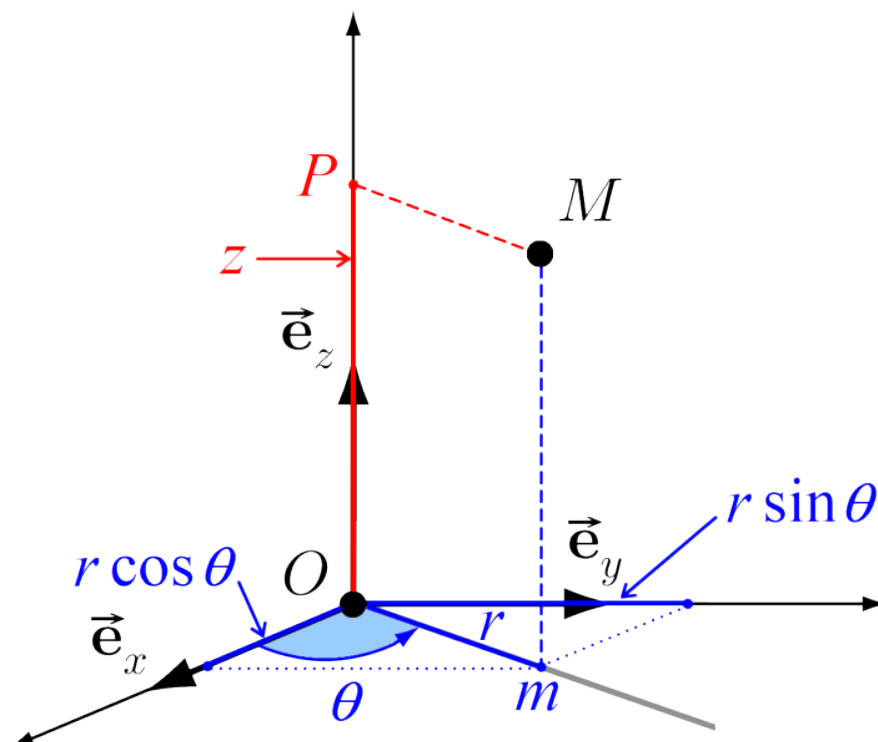
Remark • Points on (Oz) do not have unique cylindrical coordinates.

- If m and P are the projections of M onto the plane (Oxy) and the line (Oz) , then in cylindrical coordinates:

$$M(r, \theta, z), \quad m(r, \theta, 0), \quad P(0, ?, z).$$

Property (Conversion between Cartesian and Cylindrical Coordinates). To convert from Cartesian coordinates to cylindrical coordinates and vice versa:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \iff \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \quad (x \neq 0) \\ z = z \end{cases}$$



5.4 Spherical Coordinates

Definition (Spherical Coordinates). We locate a point M in space by:

$$\overrightarrow{OM} = \rho \cos \theta \sin \phi \vec{e}_x + \rho \sin \theta \sin \phi \vec{e}_y + \rho \cos \phi \vec{e}_z,$$

with $\phi \in [0, \pi]$, $\theta \in [0, 2\pi]$, $\rho \geq 0$.

The triple (ρ, ϕ, θ) constitutes the spherical coordinates of the point M .

Decomposing $\overrightarrow{OM}$ as before:

$$\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{OP}$$

with

$$\overrightarrow{OM} = \rho \cos(\theta) \sin(\phi) \vec{e}_x + \rho \sin(\theta) \sin(\phi) \vec{e}_y + \rho \cos(\phi) \vec{e}_z$$

where $\phi \in [0, \pi]$, $\theta \in [0, 2\pi]$.

Remark • Points on the (Oz) axis do not have unique spherical coordinates.

- There are several conventions for spherical and cylindrical coordinate notations. The choice adopted here is such that θ plays the same role in cylindrical and spherical coordinate systems. However, this is not always the case! Pay close attention to the conventions chosen.

Property (Conversion between Cartesian and Spherical Coordinates). To convert from Cartesian coordinates to spherical coordinates and vice versa:

$$\begin{cases} x = \rho \cos \theta \sin \phi \\ y = \rho \sin \theta \sin \phi \\ z = \rho \cos \phi \end{cases} \iff \begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \phi = \arccos\left(\frac{z}{\rho}\right) \\ \theta = \arctan\left(\frac{y}{x}\right) \quad (x \neq 0) \end{cases}$$

