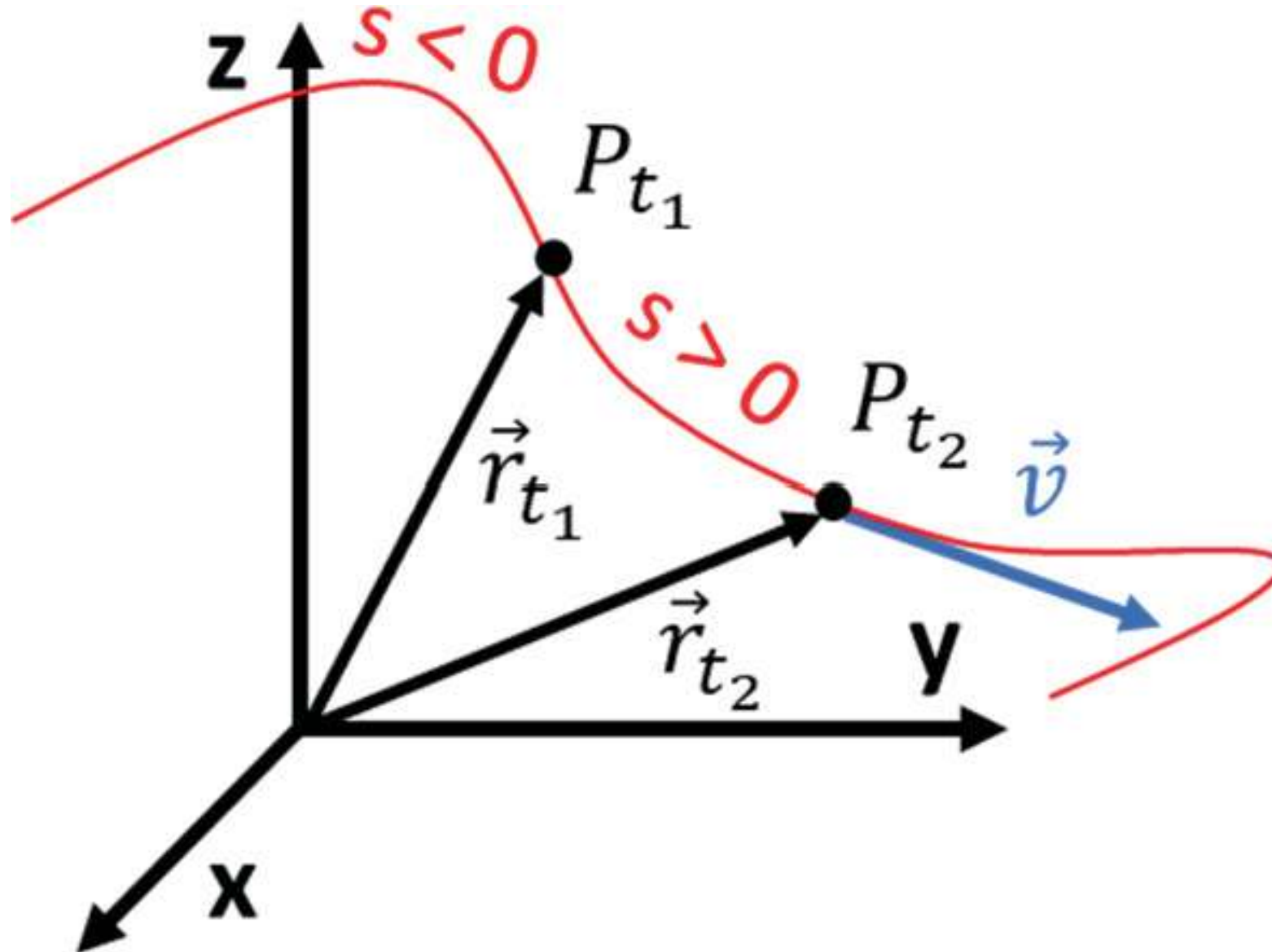


Kinematics of a Point



1. Chapter Objectives

- The concepts of **velocity** and **acceleration** are **well known**, at least intuitively.
- As such, reading this chapter does not require insurmountable effort to understand. In this chapter:
- The expressions for the **displacement**, **velocity**, and **acceleration vectors** in the main **coordinate systems** are detailed.
- Examples of simple **motions** are then studied.

2. Where and When

- To describe the **motion** of bodies, one must first choose a **reference system** or **frame of reference**, that is, one or more bodies that will conventionally be considered as fixed, to which a coordinate system will be related.
- A **frame of reference** consists of a **spatial** reference frame associated with a **temporal** reference frame;
- The kinematic study of the motion of a point amounts to being able to answer the questions "**where?**" (**space**) and "**when?**" (**time**).
- To answer these questions, it is necessary to define a spatial reference frame and a time reference frame.

2.1. Time reference frame

- It is an oriented axis indicating the direction in which time flows, equipped with an arbitrary origin ($t = 0$) and a time scale provided by a **clock**. Measuring durations requires a unit of time: the second of the International System (SI).



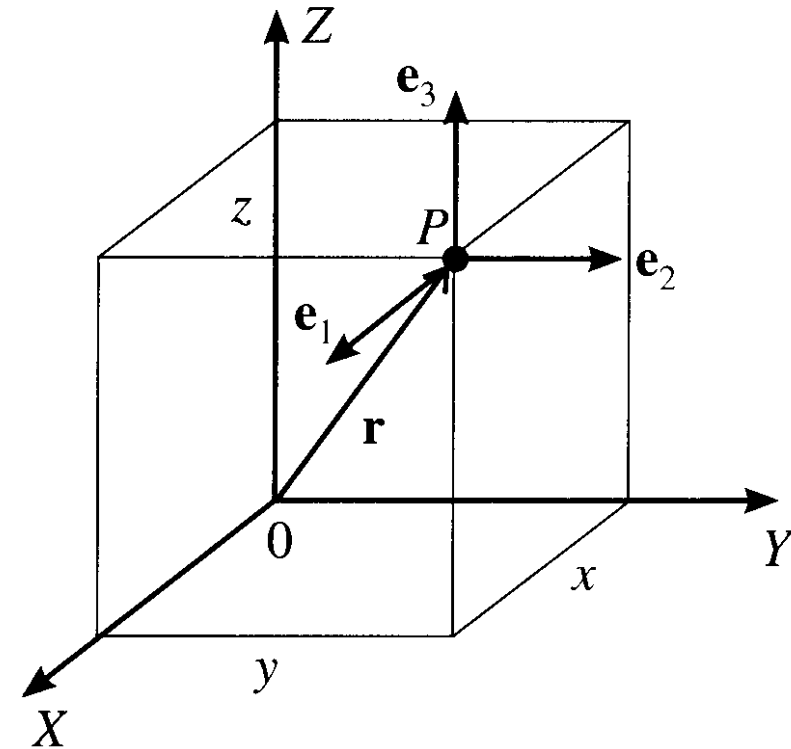
- In **classical** or **Newtonian mechanics**, it is postulated that the **time frame** is the **same** for **all** reference frames, and that time flows in the same way in reference frames moving relative to one another.

2.3. Spatial reference frame

- **Locating** a point will **allow** us to calculate its velocity and acceleration, and to describe its motion. To locate a point, we use a **reference frame**.
- A reference frame consists of an origin ***O*** and a basis **$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$** , generally orthonormal and direct (right-handed).

$$\text{Orthonormal} \Leftrightarrow \begin{cases} \vec{e}_1 \cdot \vec{e}_2 = 0 \\ \vec{e}_2 \cdot \vec{e}_3 = 0 \\ \vec{e}_1 \cdot \vec{e}_3 = 0 \end{cases} \text{ and } \{\|\vec{e}_1\| = \|\vec{e}_2\| = \|\vec{e}_3\| = 1\}$$

$$\text{Direct} \Leftrightarrow \begin{cases} \vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3 \\ \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_1 \\ \vec{e}_3 \wedge \vec{e}_1 = \vec{e}_2 \end{cases}$$



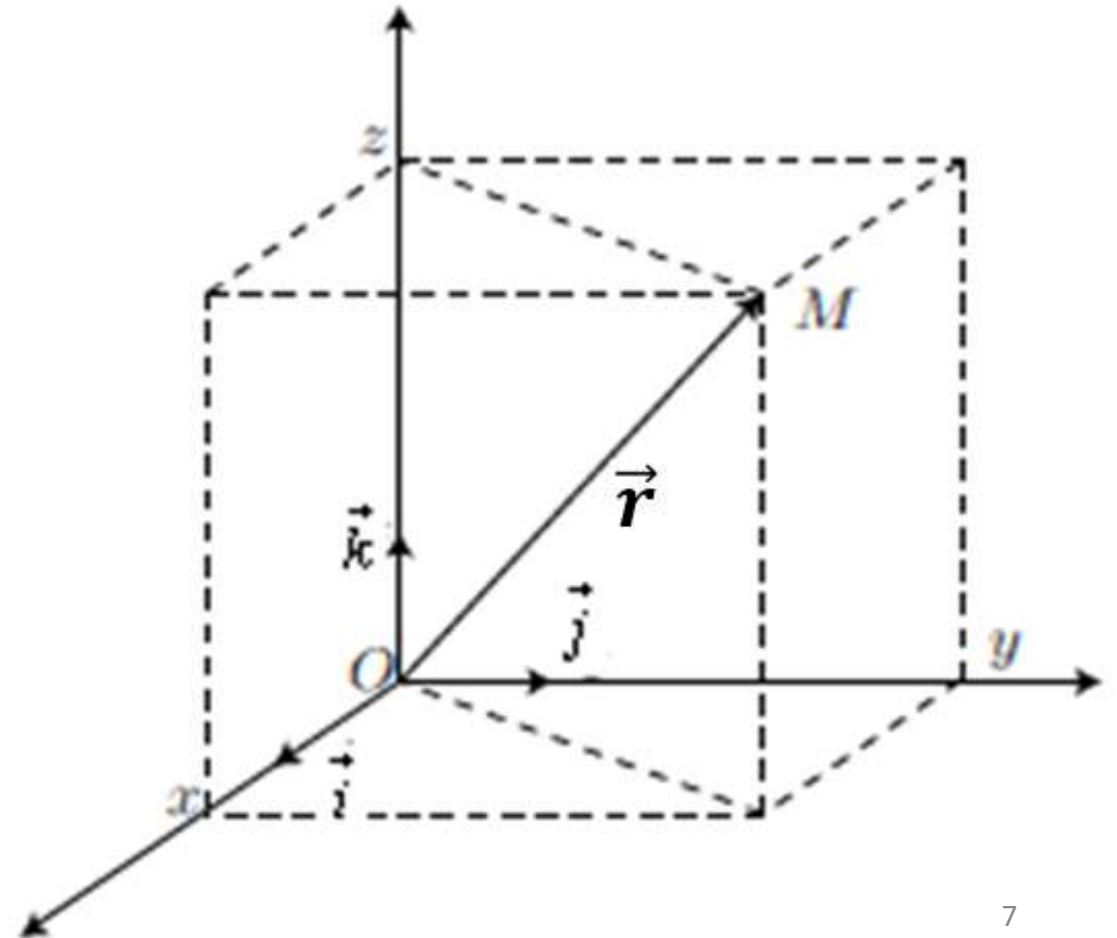
3. The kinematics

- **Kinematics** (from the Greek "kinesis," meaning **motion**) is the branch of mechanics that **studies the motion** of bodies in relation to **time**, **independently** of the causes of that motion.
- A kinematics problem consists of determining **relationships** between **lengths**, **velocities**, **accelerations**, and **time**, relating to the motion of **points** and bodies, which are assumed here to be non-deformable (rigid).
- A **particle** is an object whose position can be described by a single **point**.
Anything can be considered to be a particle – a molecule, a person or a galaxy – as long as we can reasonably **ignore its internal structure**.

4. **Position vector**, trajectory and displacement vector

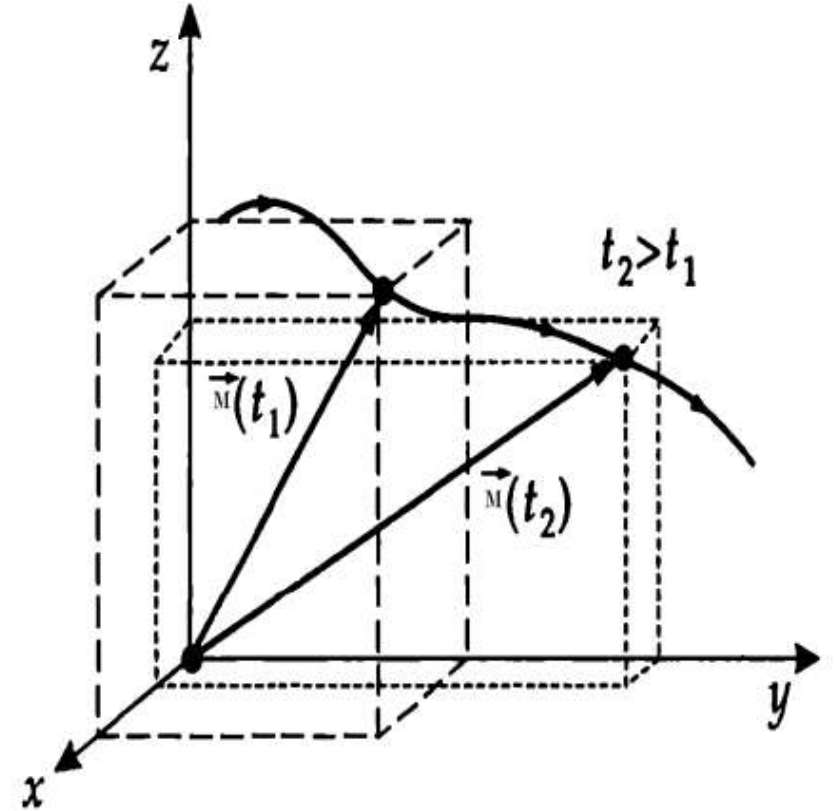
- The **position vector** of a particle is a vector drawn from the **origin** of some coordinate system to the particle located in **M** .
- The position vector $(\overrightarrow{OM})$ or $\vec{r}$ in Cartesian coordinates is:

$$\overrightarrow{OM} = \vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$$



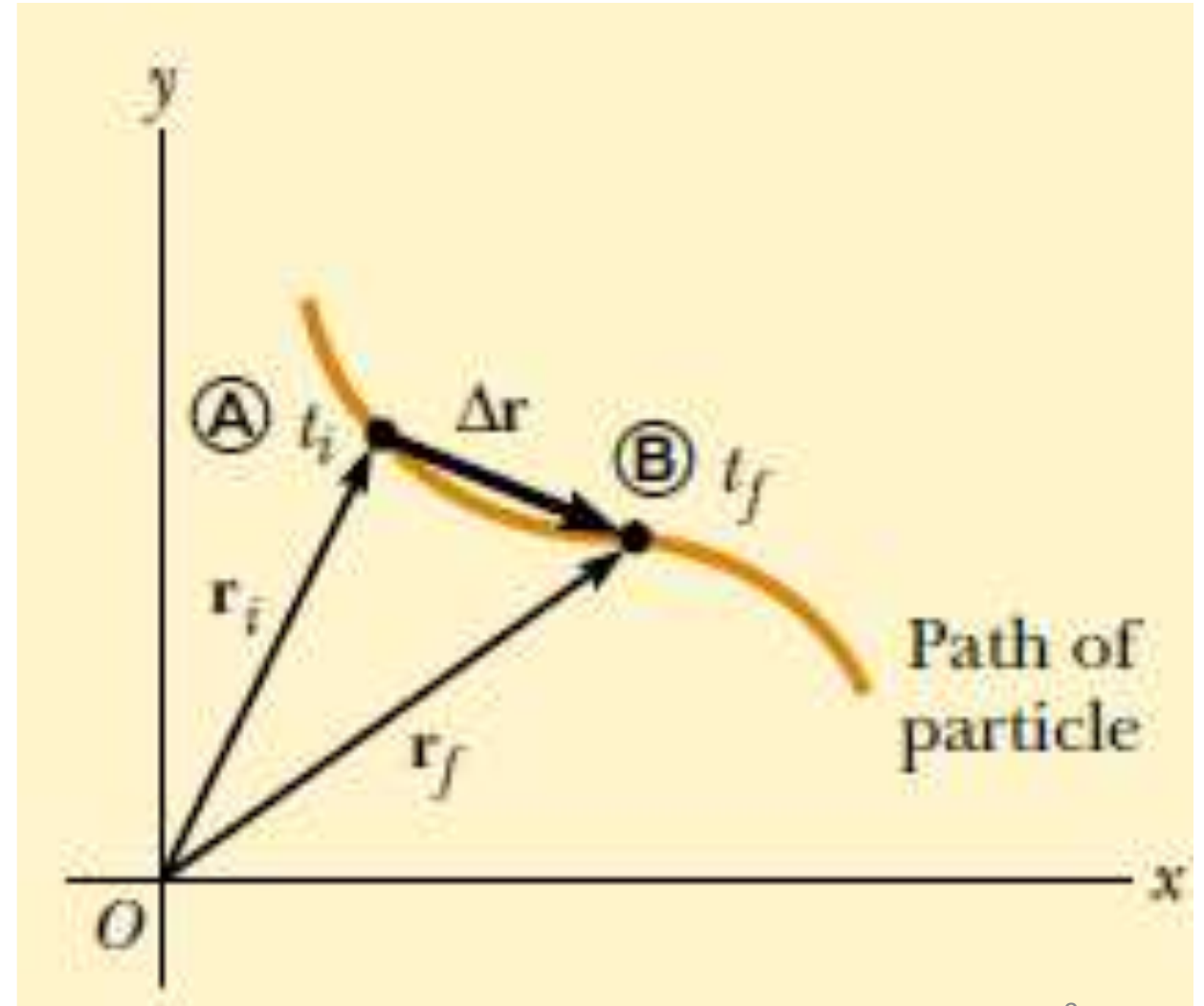
4.1. Position vector, **trajectory** and displacement vector

- The **trajectory** is the set of all points successively occupied by the particle during its motion.
- These points generally lie on a **curve**.
- The equation that gives the position of a particle at any instant t is called **the time equation of motion**.



4.2. Position vector, trajectory and **displacement** vector

- The **displacement** of a particle is defined as its **change** in **position** in some **time interval**.
- At time t_i the particle is at point **(A)**, described by position vector $\vec{r}_i$.
- At some later time t_f it is at point **(B)**, described by position vector $\vec{r}_f$.

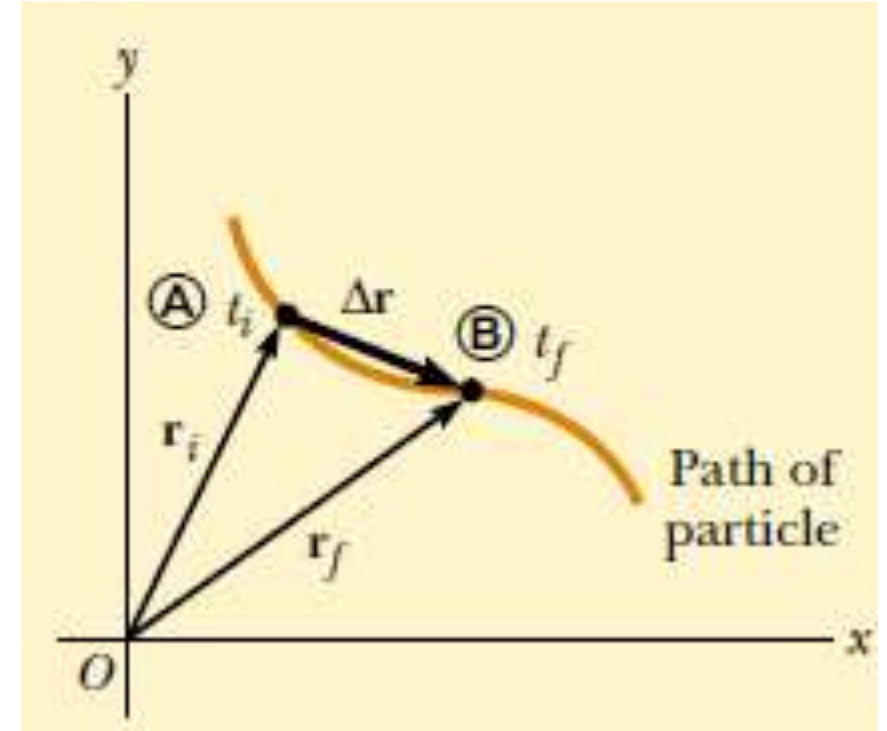


➤ The path from (A) to (B) is not necessarily a straight line.

➤ As the particle moves from (A) to (B)

in the time interval $\Delta t = t_f - t_i$,

its position vector changes from $\vec{r}_i$ to $\vec{r}_f$.



➤ We now define the displacement vector $\Delta \vec{r}$ for the particle as the difference between its final position vector and its initial position vector:

$$\Delta \vec{r} = \vec{r}_f - \vec{r}_i = \Delta x \vec{i} + \Delta y \vec{j} + \Delta z \vec{k}$$

➤ The definition of the "elementary displacement vector" $\overrightarrow{dl}$ is similar to that of the "displacement vector", but this time the times t_i and t_f are very close, such that $t_f - t_i \rightarrow 0$. We then have: $\overrightarrow{dl} = \lim_{\Delta t \rightarrow 0} (\overrightarrow{r_f} - \overrightarrow{r_i}) = \lim_{\Delta t \rightarrow 0} (\Delta \overrightarrow{r})$

➤ Mathematically, this amounts to differentiating the expression of $(\overrightarrow{OM})$ or $\overrightarrow{r}$. It comes : $\overrightarrow{dl} = d(\overrightarrow{r}) = d(x \vec{i} + y \vec{j} + z \vec{k})$

$$\overrightarrow{dl} = (dx \vec{i} + x d\vec{i}) + (dy \vec{j} + y d\vec{j}) + (dz \vec{k} + z d\vec{k})$$

➤ In the Cartesian coordinate system, the basis vectors are fixed, that is to say

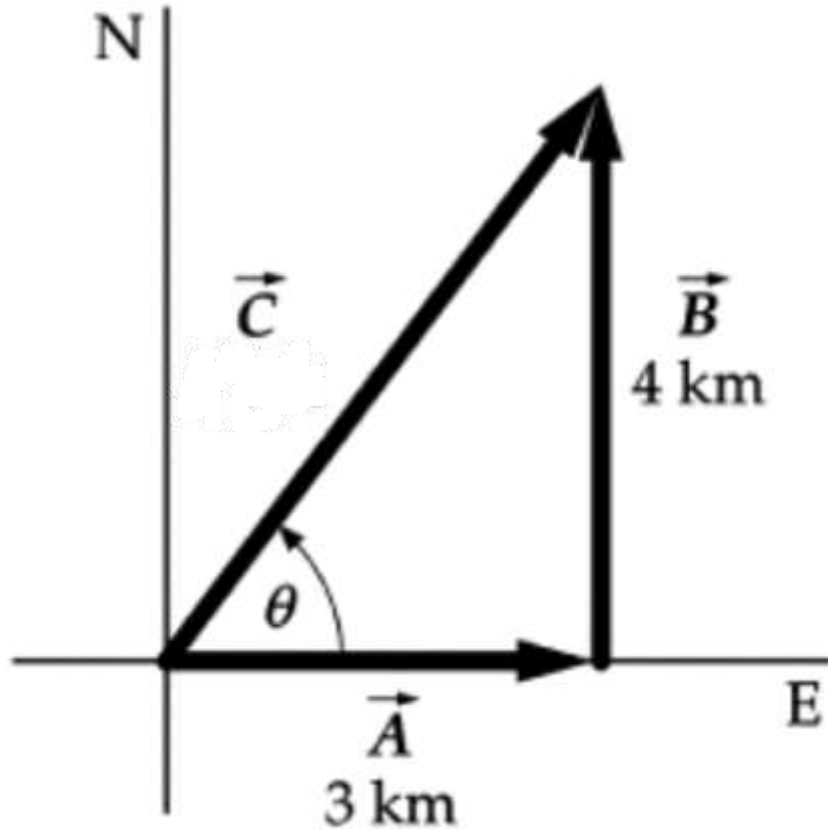
$$d\vec{i} = d\vec{j} = d\vec{k} = 0$$

finally obtained:

$$\boxed{\overrightarrow{dl} = dx \vec{i} + dy \vec{j} + dz \vec{k}}$$

Example 1

You walk **3 km** east and then **4 km** north.



What is your resultant displacement?

5. Velocity and acceleration vectors

- We consider that velocity (unit: $m \cdot s^{-1}$) is the distance traveled per unit of time.
- We consider that acceleration (unit: $m \cdot s^{-2}$) is a measure of the variation of velocity per unit of time.
- It is also the ratio of the distance traveled to the square of the time taken to travel it.

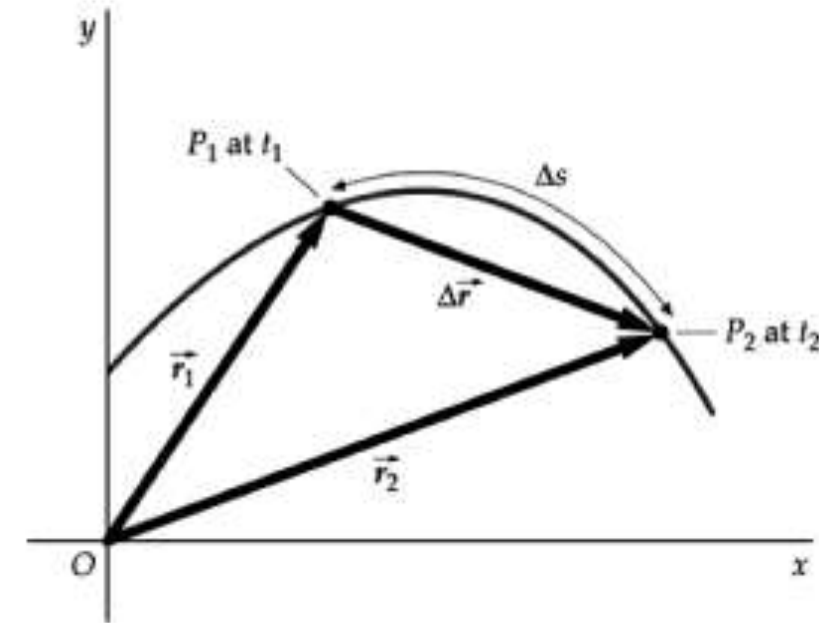
5.1. Average-velocity vector $\vec{v}_{av}$

➤ The ratio of the displacement vector to the time interval $\Delta t = t_2 - t_1$ is the average velocity vector:

$$\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t}$$

➤ This vector points in the direction of the displacement.

➤ The magnitude of the displacement vector is less than the distance traveled along the curve unless the particle moves in a straight line.



➤ In Cartesian coordinates :

$$\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t} \vec{i} + \frac{\Delta y}{\Delta t} \vec{j} + \frac{\Delta z}{\Delta t} \vec{k}$$

$$\|\vec{v}_{av}\| = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}$$

Example 2

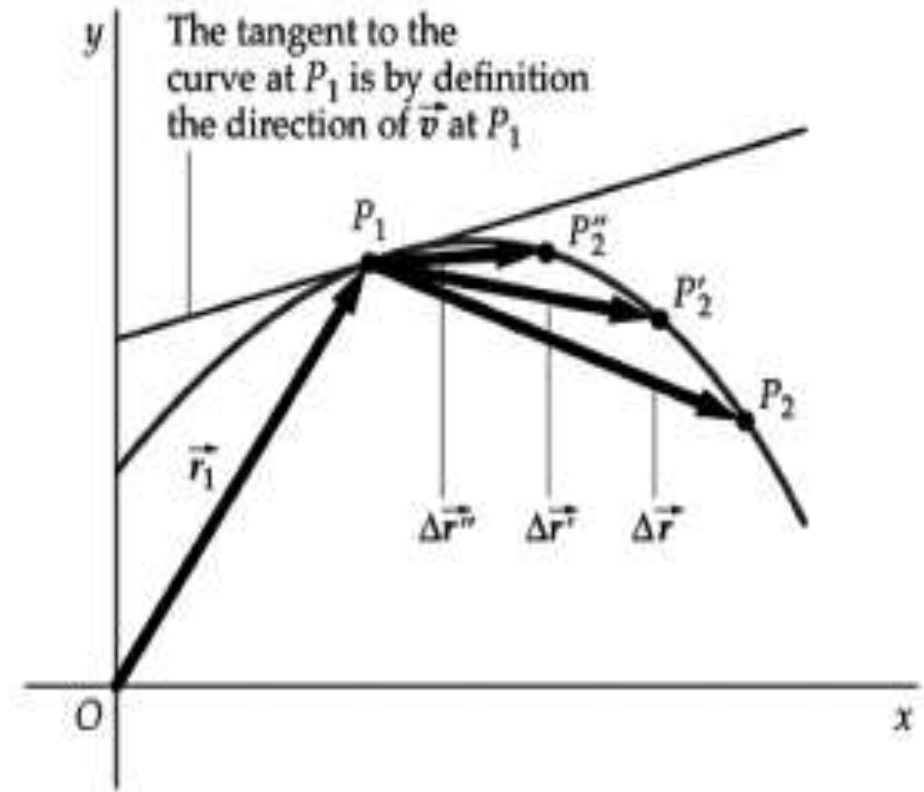
The position of a pinewood derby car was observed at various times; the results are summarized in the following table :

$t \text{ (s)}$	0,0	1,0	2,0	3,0	4,0	5,0
$x \text{ (m)}$	0,0	2,3	9,2	20,7	36,8	57,5

Find the average velocity of the car for (a) the first second, (b) the last 3 s, and (c) the entire period of observation.

5.2. Instantaneous-velocity vector $\vec{v}$

- If we consider smaller and smaller intervals,
- the magnitude of the displacement approaches the distance along the curve, and the direction of approaches the tangent to the curve at the beginning of the interval.



- We define the instantaneous-velocity vector as the limit of the average-velocity vector as Δt approaches zero:

$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

- The instantaneous-velocity vector is the derivative of the position vector with respect to time.
- In Cartesian coordinates :

$$\left\{ \begin{array}{l} \vec{v} = \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} + \frac{dz}{dt} \vec{k} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k} \\ \|\vec{v}\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(v_x)^2 + (v_y)^2 + (v_z)^2} \end{array} \right.$$
- Its magnitude is the **speed** $\left(\frac{ds}{dt}\right)$, and its direction is the direction of motion of the particle along the line tangent to the curve.

Example 3

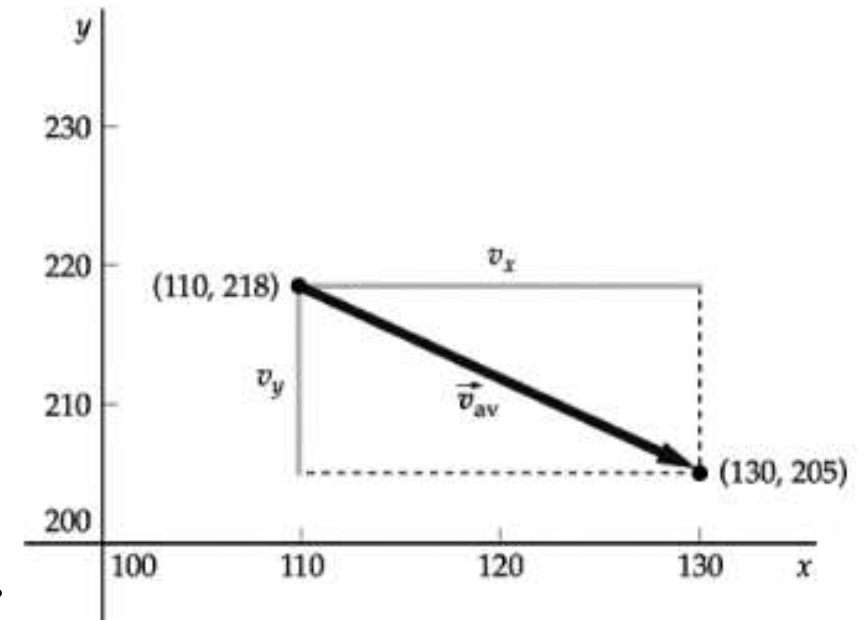
A particle moves along the x -axis in such a way that its position at any instant is given by : $x = 5 t^2 + 1$, where x is in meters and t is in seconds.

Compute its average velocity in the time interval between :

- (a) 2 s and 3 s ,
- (b) 2 s and 2.1 s ,
- (c) 2 s and 2.001 s ,
- (d) 2 s and 2.00001 s
- (e) Also compute the instantaneous velocity at 2 s

Example 4

A sailboat has coordinates $(x_1, y_1) = (110 \text{ m}, 218 \text{ m})$ at $t_1 = 60 \text{ s}$. Two minutes later, at time t_2 , it has the coordinates $(x_2, y_2) = (130 \text{ m}, 205 \text{ m})$.



- a) Find the average velocity for this two-minute interval.

Express $\vec{v}_{av}$ in terms of its rectangular components.

- b) Find the magnitude and direction of this average velocity.

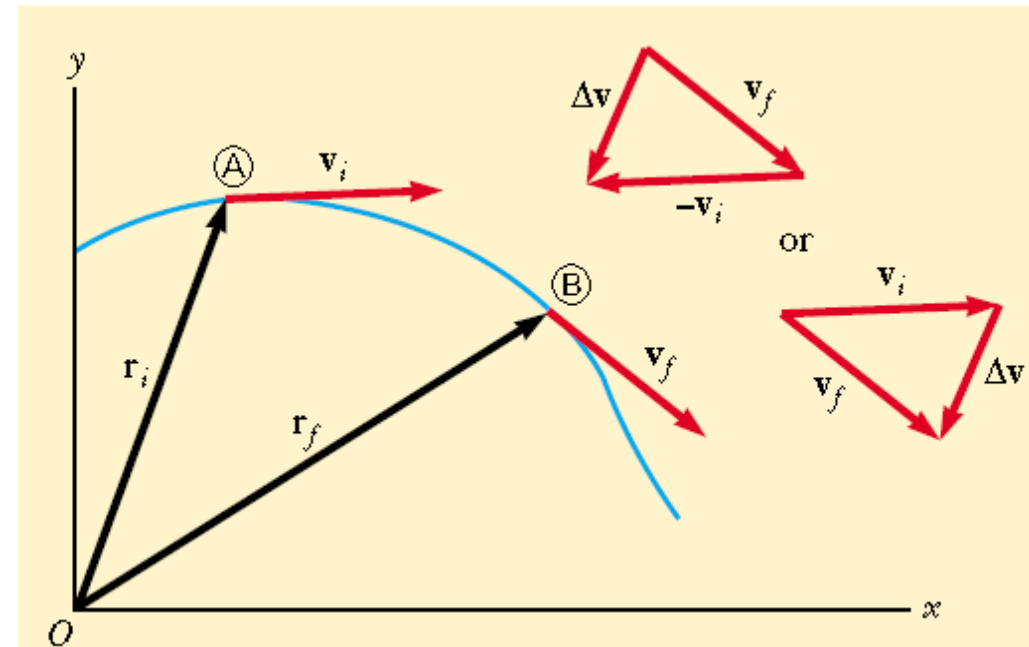
- c) For $t \geq 20 \text{ s}$, the position of the sailboat as a function of time is

$$x(t) = 100 \text{ m} + \left(\frac{1}{6} \text{ ms}^{-1}\right) t \text{ and } y(t) = 200 \text{ m} + (1080 \text{ m} \cdot \text{s}) t^{-1}.$$

Find the instantaneous velocity at a general time $t \geq 20 \text{ s}$.

5.3. The Average acceleration vector $\vec{a}_{av}$

- As a particle moves from one point to another along some path, its **instantaneous velocity** vector **changes** from $\vec{v}_i$ at time t_i to $\vec{v}_f$ at time t_f .
- Knowing the velocity at these points allows us to determine the **average acceleration** of the particle.
- The average acceleration $\vec{a}_{av}$ of a particle as it moves is defined as the **change** in the **instantaneous velocity** vector $\Delta\vec{v}$ **divided by** the time interval Δt during which that change occurs:
$$\vec{a}_{av} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{\Delta\vec{v}}{\Delta t}$$
- Because $\vec{a}_{av}$ is the ratio of a vector quantity $\Delta\vec{v}$ and a positive scalar quantity Δt , we conclude that average acceleration is a vector quantity directed along $\Delta\vec{v}$.



Example 5

A car is traveling east at $60 \frac{\text{km}}{\text{h}}$. It rounds a curve, and 5 s later it is traveling north at $60 \frac{\text{km}}{\text{h}}$.

- 1) Find the average acceleration of the car.
- 2) Find the magnitude and direction of the average acceleration vector.

5.4. The instantaneous acceleration $\vec{a}$

- When the average acceleration of a particle changes during different time intervals, it is useful to define its instantaneous acceleration.
- The instantaneous acceleration $\vec{a}$ is defined as the limiting value of the ratio $\frac{\Delta \vec{v}}{\Delta t}$ as Δt approaches zero:

$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

- In other words, the **instantaneous acceleration** equals the **derivative of the velocity** vector with respect to time.

To calculate the instantaneous acceleration, we express in Cartesian coordinates:

$$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k} = \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} + \frac{dz}{dt} \vec{k}$$

Then

$$\vec{a} = \frac{dv_x}{dt} \vec{i} + \frac{dv_y}{dt} \vec{j} + \frac{dv_z}{dt} \vec{k} = \frac{d^2x}{dt^2} \vec{i} + \frac{d^2y}{dt^2} \vec{j} + \frac{d^2z}{dt^2} \vec{k}$$

$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

Example 6

The position of a thrown football is given by

$$\vec{r} = 1,5 \text{ m } \vec{i} + (12 \text{ m/s } \vec{i} + 16 \text{ m/s } \vec{j})t - (4,9 \text{ m/s}^2 \vec{j})t^2$$

Find its velocity and acceleration.

Example 7

Let the position vector be:

$$\overrightarrow{OM} = \begin{pmatrix} x = 2t^2 \\ y = 4t - 5 \\ z = t^3 \end{pmatrix}$$

For both instants $t_1 = 1s$ and $t_2 = 3s$,

deduce the average and instantaneous velocity and acceleration vectors, then calculate the magnitude of each.

6. Kinematics in Cartesian, cylindrical, spherical and Frenet frames

- In this section, the formulas for the **position**, **velocity**, and **acceleration** of a material point in **Cartesian**, **cylindrical**, **spherical**, and **Frenet frames** will be established.
- These formulas are general: that is, they are capable of perfectly describing all possible trajectories, positions, velocities, or accelerations.

6.1. Kinematics in the Cartesian coordinate system

In a Cartesian coordinate system, the position vector ($\vec{r}$ or $\overrightarrow{OM}$), elementary displacement vector ($d\vec{l}$), velocity vector ($\vec{V}$), and acceleration vector ($\vec{a}$) are given by the following expressions:

$$\vec{r} = \overrightarrow{OM} = x \vec{i} + y \vec{j} + z \vec{k}$$

$$d\vec{l} = dx \vec{i} + dy \vec{j} + dz \vec{k}$$

$$\vec{V} = \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} + \frac{dz}{dt} \vec{k} = \dot{x} \vec{i} + \dot{y} \vec{j} + \dot{z} \vec{k}$$

$$\vec{a} = \frac{d^2x}{dt^2} \vec{i} + \frac{d^2y}{dt^2} \vec{j} + \frac{d^2z}{dt^2} \vec{k} = \ddot{x} \vec{i} + \ddot{y} \vec{j} + \ddot{z} \vec{k}$$

6.2. Kinematics in cylindrical and polar frames

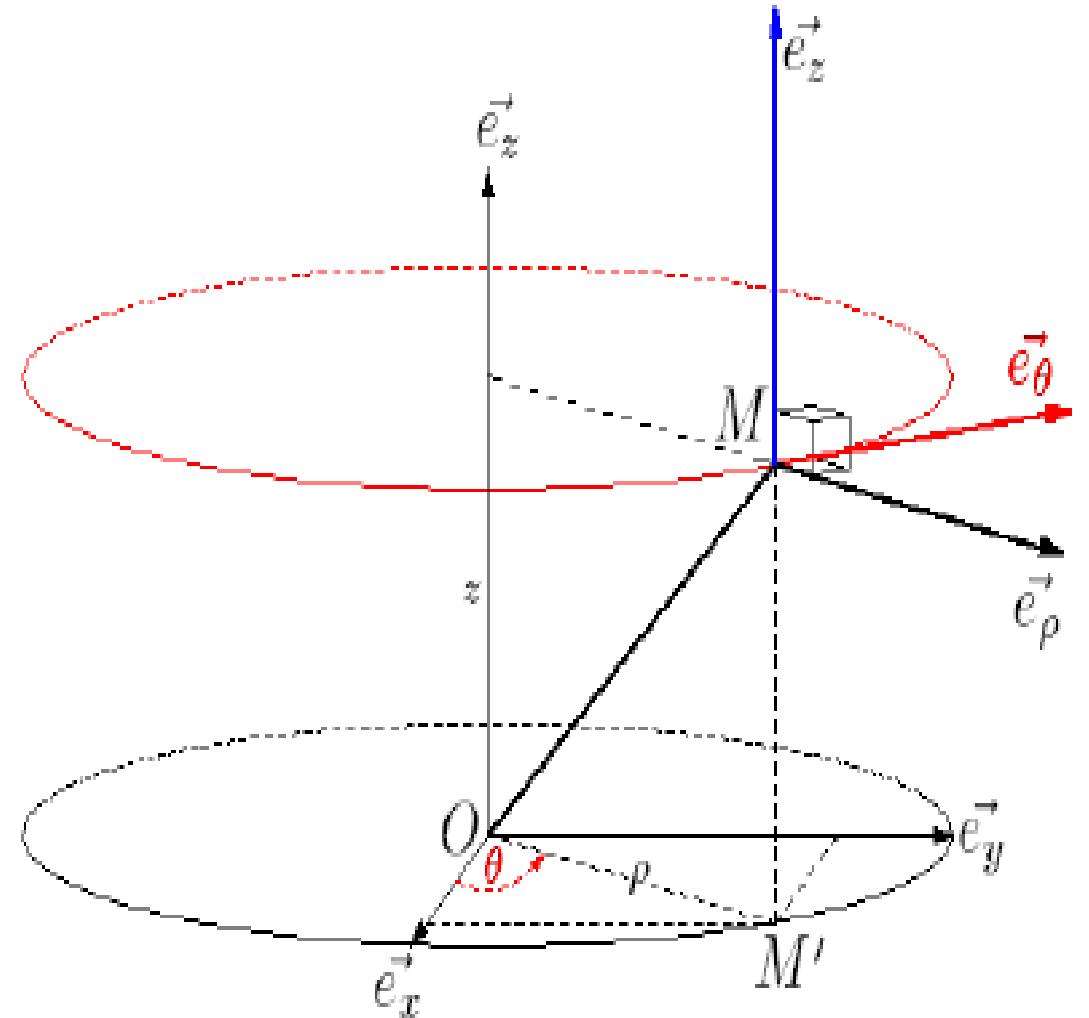
In cylindrical coordinates, the position vector is written:

$$\vec{r} = \overrightarrow{OM} = \rho \vec{e}_\rho + z \vec{e}_z$$

The elementary displacement vector $d\vec{l}$ is:

$$d\vec{l} = d\rho \vec{e}_\rho + \rho d\theta \vec{e}_\theta + dz \vec{e}_z$$

$$d\vec{l} = \begin{pmatrix} d\rho \\ \rho d\theta \\ dz \end{pmatrix}$$



The velocity vector is derived from the elementary displacement vector $\vec{v} = \frac{d\vec{l}}{dt}$

$$\vec{v} = \frac{d\rho}{dt} \vec{e}_\rho + \rho \frac{d\theta}{dt} \vec{e}_\theta + \frac{dz}{dt} \vec{e}_z \quad \text{or} \quad \vec{v} = \dot{\rho} \vec{e}_\rho + \rho \dot{\theta} \vec{e}_\theta + \dot{z} \vec{e}_z \quad \text{or} \quad \vec{v} = \begin{pmatrix} \dot{\rho} \\ \rho \dot{\theta} \\ \dot{z} \end{pmatrix}$$

The acceleration vector can be derived from the velocity vector $\vec{a} = \frac{d\vec{v}}{dt}$

$$\begin{aligned} \vec{a} &= \frac{d}{dt} \left[\frac{d\rho}{dt} \vec{e}_\rho + \rho \frac{d\theta}{dt} \vec{e}_\theta + \frac{dz}{dt} \vec{e}_z \right] \\ \vec{a} &= \left[\frac{d^2\rho}{dt^2} \vec{e}_\rho + \frac{d\rho}{dt} \frac{d\vec{e}_\rho}{dt} \right] + \left[\frac{d\rho}{dt} \frac{d\theta}{dt} \vec{e}_\theta + \rho \frac{d}{dt} \left(\frac{d\theta}{dt} \vec{e}_\theta \right) \right] + \left[\frac{d^2z}{dt^2} \vec{e}_z + \frac{dz}{dt} \frac{d\vec{e}_z}{dt} \right] \\ \vec{a} &= \left[\frac{d^2\rho}{dt^2} \vec{e}_\rho + \frac{d\rho}{dt} \frac{d\vec{e}_\rho}{dt} \right] + \left[\frac{d\rho}{dt} \frac{d\theta}{dt} \vec{e}_\theta + \rho \frac{d^2\theta}{dt^2} \vec{e}_\theta + \rho \frac{d\theta}{dt} \frac{d\vec{e}_\theta}{dt} \right] + \left[\frac{d^2z}{dt^2} \vec{e}_z \right] \\ \vec{a} &= \left[\frac{d^2\rho}{dt^2} \vec{e}_\rho + \frac{d\rho}{dt} \frac{d\theta}{dt} \vec{e}_\theta \right] + \left[\frac{d\rho}{dt} \frac{d\theta}{dt} \vec{e}_\theta + \rho \frac{d^2\theta}{dt^2} \vec{e}_\theta - \rho \frac{d\theta}{dt} \frac{d\theta}{dt} \vec{e}_\rho \right] + \left[\frac{d^2z}{dt^2} \vec{e}_z \right] \\ \vec{a} &= \left[\frac{d^2\rho}{dt^2} - \rho \left(\frac{d\theta}{dt} \right)^2 \right] \vec{e}_\rho + \left[2 \frac{d\rho}{dt} \frac{d\theta}{dt} + \rho \frac{d^2\theta}{dt^2} \right] \vec{e}_\theta + \left[\frac{d^2z}{dt^2} \right] \vec{e}_z \end{aligned}$$

This last equation is the general expression of acceleration in the cylindrical reference frame. In simplified notation, it is also written:

$$\vec{a} = [\rho'' - \rho\theta'^2]\vec{e}_\rho + [2\rho'\theta' + \rho\theta'']\vec{e}_\theta + [z'']\vec{e}_z \quad \text{or} \quad \vec{a} = \begin{pmatrix} \rho'' - \rho\theta'^2 \\ 2\rho'\theta' + \rho\theta'' \\ z'' \end{pmatrix}$$

Note:

If the point needs to be located in the **plane**, **polar coordinates** can be used. Simply **remove** the **Oz** axis and its coordinate from the cylindrical coordinate system to obtain polar coordinates.

The position vector $(\overrightarrow{OM})$, elementary displacement vector $(\overrightarrow{dl})$, velocity vector $(\vec{V})$ and acceleration vector $(\vec{a})$ in **polar coordinates** are given by the following expressions:

$$\vec{r} = \overrightarrow{OM} = \rho \overrightarrow{e_\rho}$$

$$d\vec{l} = d\rho \overrightarrow{e_\rho} + \rho d\theta \overrightarrow{e_\theta} \quad \text{or} \quad d\vec{l} = \begin{pmatrix} d\rho \\ \rho d\theta \end{pmatrix}$$

$$\vec{v} = \frac{d\rho}{dt} \overrightarrow{e_\rho} + \rho \frac{d\theta}{dt} \overrightarrow{e_\theta} \quad \text{or} \quad \vec{v} = \dot{\rho} \overrightarrow{e_\rho} + \rho \dot{\theta} \overrightarrow{e_\theta} \quad \text{or} \quad \vec{v} = \begin{pmatrix} \dot{\rho} \\ \rho \dot{\theta} \end{pmatrix}$$

$$\vec{a} = [\rho'' - \rho \theta'^2] \overrightarrow{e_\rho} + [2\rho' \theta' + \rho \theta''] \overrightarrow{e_\theta} \quad \text{or} \quad \vec{a} = \begin{pmatrix} \rho'' - \rho \theta'^2 \\ 2\rho' \theta' + \rho \theta'' \end{pmatrix}$$

Example 8

The polar coordinates of a point are $r = 5,50\text{ m}$ and $\theta = 240^\circ$.

What are the Cartesian coordinates of this point?

Example 9

Two points in a plane have polar coordinates $(2.50\text{ m}, 30.0^\circ)$ and $(3.80\text{ m}, 120.0^\circ)$.

Determine

- (a) the Cartesian coordinates of these points
- (b) the distance between them.

Example 10

Two points in the xy plane have Cartesian coordinates $(2.00, -4.00) \text{ m}$ and $(-3.00, 3.00) \text{ m}$.

Determine :

- (a) The distance between these points
- (b) their polar coordinates.

6.3. Kinematics in the spherical frame of reference

In spherical coordinates, the position vector is written:

$$\vec{r} = \overrightarrow{OM} = r \vec{u}_r$$

The elementary displacement vector $d\vec{l}$ is:

$$d\vec{l} = r \vec{u}_r + r d\theta \vec{u}_\theta + r d\varphi \sin \theta \vec{u}_\varphi$$

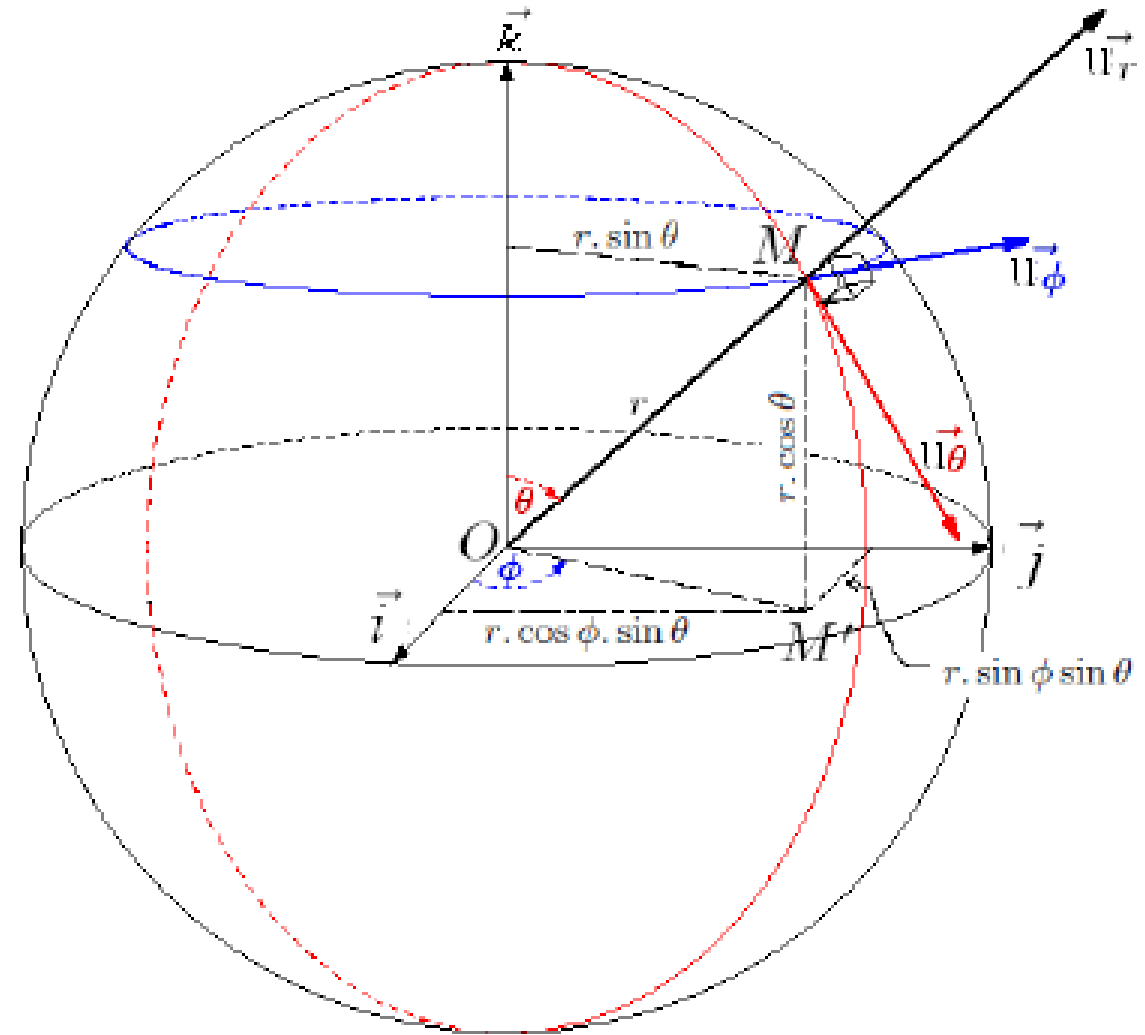
The velocity vector is derived from the

elementary displacement vector : $\vec{v} = \frac{d\vec{l}}{dt}$

$$\vec{v} = \frac{dr}{dt} \vec{u}_r + r \frac{d\theta}{dt} \vec{u}_\theta + r \frac{d\varphi}{dt} \sin \theta \vec{u}_\varphi$$

$$\vec{v} = r' \vec{u}_r + r \theta' \vec{u}_\theta + r \varphi' \sin \theta \vec{u}_\varphi$$

$$\vec{v} = \begin{pmatrix} r' \\ r \theta' \\ r \varphi' \sin \theta \end{pmatrix}$$



The acceleration vector can be derived from the velocity vector $\vec{a} = \frac{d\vec{v}}{dt}$

$$\vec{a} = \frac{d}{dt} \left[\frac{dr}{dt} \vec{u}_r + r \frac{d\theta}{dt} \vec{u}_\theta + r \frac{d\varphi}{dt} \sin \theta \vec{u}_\varphi \right]$$

$$\vec{a} = \frac{d}{dt} \left(\frac{dr}{dt} \vec{u}_r \right) + \frac{d}{dt} \left(r \frac{d\theta}{dt} \vec{u}_\theta \right) + \frac{d}{dt} \left(r \frac{d\varphi}{dt} \sin \theta \vec{u}_\varphi \right)$$

$$\vec{a} = \left(r'' \vec{u}_r + r' \frac{d\vec{u}_r}{dt} \right) + \left(r' \theta' \vec{u}_\theta + r \frac{d}{dt} (\theta' \vec{u}_\theta) \right) + \left(\frac{d}{dt} (r \sin \theta) \varphi' \vec{u}_\varphi + r \sin \theta \frac{d}{dt} (\varphi' \vec{u}_\varphi) \right)$$

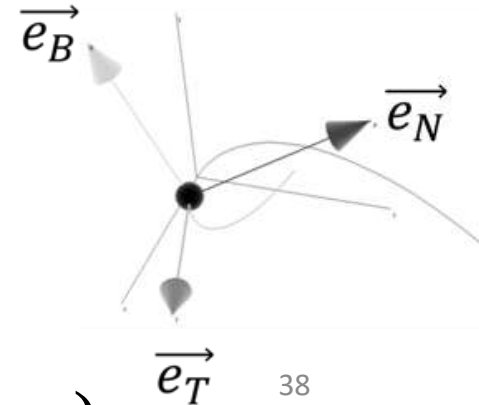
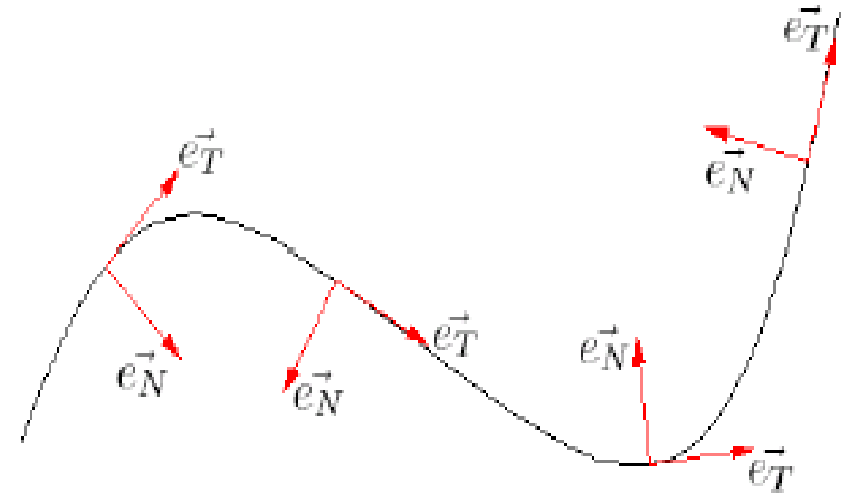
$$\vec{a} = \left(r'' \vec{u}_r + r' \frac{d\vec{u}_r}{dt} \right) + \left(r' \theta' \vec{u}_\theta + r \left(\theta'' \vec{u}_\theta + \theta' \frac{d\vec{u}_\theta}{dt} \right) \right) + \left((r' \sin \theta + r \theta' \cos \theta) \varphi' \vec{u}_\varphi + r \sin \theta \left(\varphi'' \vec{u}_\varphi + \varphi' \frac{d\vec{u}_\varphi}{dt} \right) \right)$$

$$\vec{a} = \left(r'' \vec{u}_r + r' \frac{d\vec{u}_r}{dt} \right) + \left(r' \theta' \vec{u}_\theta + r \theta'' \vec{u}_\theta + r \theta' \frac{d\vec{u}_\theta}{dt} \right) + \left(r' \sin \theta \varphi' \vec{u}_\varphi + r \theta' \varphi' \cos \theta \vec{u}_\varphi + \right.$$

$$\vec{a} = \left(r'' \vec{u}_r + r' (\theta' \vec{u}_\theta + \varphi' \sin \theta \vec{u}_\varphi) \right) + \left(r' \theta' \vec{u}_\theta + r \theta'' \vec{u}_\theta + r \theta' (-\theta' \vec{u}_r + \varphi' \cos \theta \vec{u}_\varphi) \right) + \left(r' \sin \theta \varphi' \vec{u}_\varphi + \right.$$

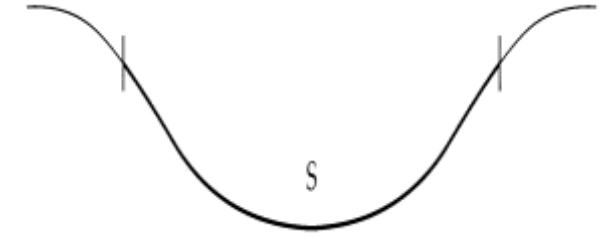
6.4. Kinematics in the Frenet frame: The Moving Trihedron

- The Frenet frame is a reference frame that can be inscribed within the trajectory of a curve, following the same point that will be its origin.
 - We define the unit vector $\vec{e}_T$ as the unit vector tangent to the curve, oriented in the direction of motion.
 - We introduce the vector $\vec{e}_N$, orthogonal to $\vec{e}_T$ and directed along the concavity of the trajectory.
 - We complete these two unit vectors with the vector $\vec{e}_B$, $\vec{e}_B = \vec{e}_T \wedge \vec{e}_N$. The vector thus defined is called the binormal.
- The basis $(\vec{e}_T, \vec{e}_N, \vec{e}_B)$ constitutes the Frenet trihedron (or Frenet frame).



➤ In the Frenet frame, it is impossible to explicitly position write the vector $\overrightarrow{OM}$.

➤ However, we define a curvilinear abscissa s (a scalar) trajectory, along the equal to the arc length $\widehat{OM}$.



➤ The elementary displacement vector is simply written as: $d\overrightarrow{OM} = d\vec{l} = ds \overrightarrow{e_T}$.

➤ This equation also allows us to give a new definition of the vector $\overrightarrow{e_T} : \overrightarrow{e_T} = \frac{d\vec{l}}{ds}$

➤ The velocity vector is derived from that of the elementary displacement vector:

$$\vec{v} = \frac{ds}{dt} \overrightarrow{e_T}$$

➤ The velocity vector is always directed along the vector $\overrightarrow{e_T}$. Naturally, it follows that:

$$\vec{v} = v \overrightarrow{e_T}$$

The acceleration vector can be derived from the velocity vector $\vec{a} = \frac{d\vec{v}}{dt}$

$$\vec{a} = \frac{d}{dt} (v \vec{e}_T)$$

$$\vec{a} = \frac{dv}{dt} \vec{e}_T + v \frac{d\vec{e}_T}{dt} \quad ; \frac{d\vec{e}_T}{dt} = ?$$

The derivative $\left(\frac{d\vec{e}_T}{dt}\right)$ is always orthogonal to $\vec{e}_T$ because:

$$\vec{e}_T \cdot \vec{e}_T = 1$$

$$\frac{d}{dt} (\vec{e}_T \cdot \vec{e}_T) = \frac{d\vec{e}_T}{dt} \cdot \vec{e}_T + \vec{e}_T \cdot \frac{d\vec{e}_T}{dt} = 0$$

$$\frac{d}{dt} (\vec{e}_T \cdot \vec{e}_T) = 2\vec{e}_T \cdot \frac{d\vec{e}_T}{dt} = 0$$

$$\vec{e}_T \cdot \frac{d\vec{e}_T}{dt} = 0$$

This means that the derivative $\frac{d\vec{e}_T}{dt}$ is always orthogonal to $\vec{e}_T$.

Suppose that the vector $\vec{e}_T$ makes an angle $\theta(t)$ with the $x - axis$. Then :

$$\vec{e}_T = \cos \theta(t) \vec{i} + \sin \theta(t) \vec{j}$$

$$\vec{e}_N = -\sin \theta(t) \vec{i} + \cos \theta(t) \vec{j}$$

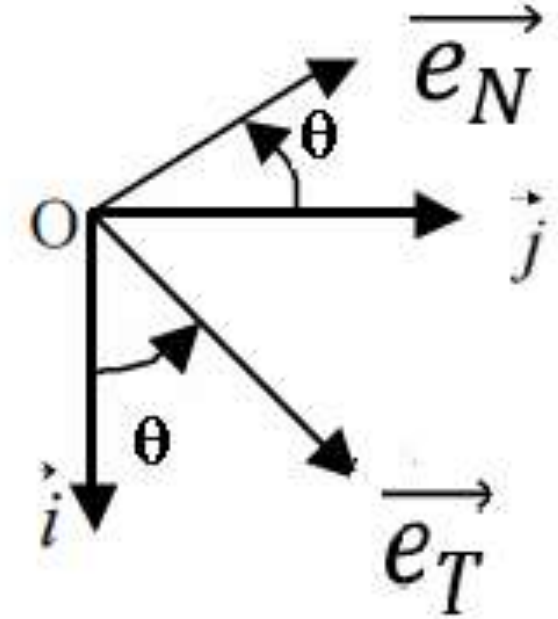
The differential of $\vec{e}_T$ is:

$$d\vec{e}_T = d\theta(-\sin \theta(t) \vec{i} + \cos \theta(t) \vec{j})$$

$$\frac{d\vec{e}_T}{dt} = \frac{d\theta}{dt} (-\sin \theta(t) \vec{i} + \cos \theta(t) \vec{j})$$

$$\frac{d\vec{e}_T}{dt} = \frac{d\theta}{dt} \vec{e}_N$$

On the other hand : $\frac{d\theta}{dt} = \frac{d\theta}{ds} \frac{ds}{dt} = v \frac{d\theta}{ds}$



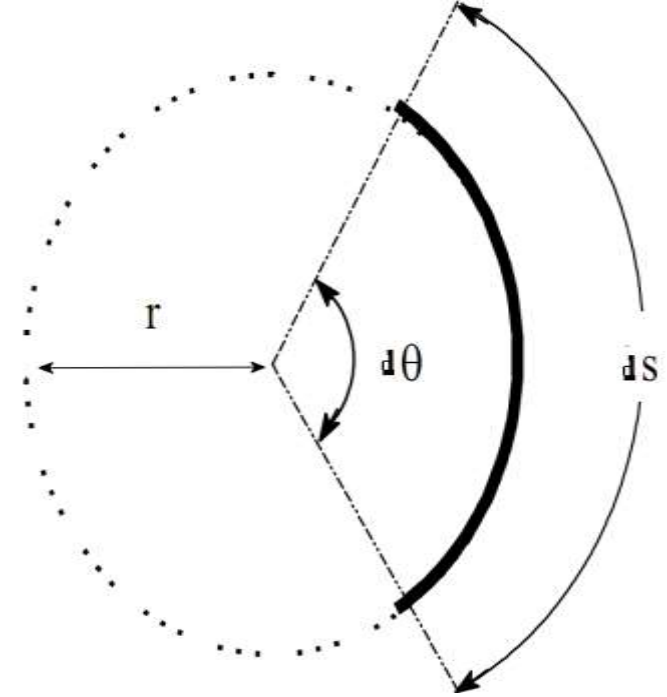
We can write :

$$ds = r d\theta \Leftrightarrow \frac{d\theta}{ds} = \frac{1}{r} \text{ so } \frac{d\theta}{dt} = \frac{v}{r} \text{ and } \boxed{\frac{d\vec{e}_T}{dt} = \frac{v}{r} \vec{e}_N}$$

(r is called the radius of curvature).

Substituting this result into the expression for $\vec{a}$,
we finally obtain :

$$\boxed{\vec{a} = \frac{dv}{dt} \vec{e}_T + \frac{v^2}{r} \vec{e}_N}$$



$\frac{dv}{dt} \vec{e}_T$: is a vector tangent to the curve, proportional to the derivative with respect to the magnitude of the velocity vector; it corresponds to the tangential acceleration $\vec{a}_T$.

$\frac{v^2}{r} \vec{e}_N$: is a normal vector to the curve, and which corresponds to the normal acceleration $\vec{a}_N$. It is associated with the change of direction, because it corresponds to $\frac{d\vec{e}_T}{dt}$.

Regarding the quantities, we can write :

$$\boxed{\vec{a} = a_T \vec{e}_T + a_N \vec{e}_N} \text{ with } a_T = \frac{dv}{dt} ; a_N = \frac{v^2}{r}$$

Example 11

A point mass, represented by a material point M , moves along a parabolic trajectory as shown in figure (1).

At time $t = 0$, the object is at O .

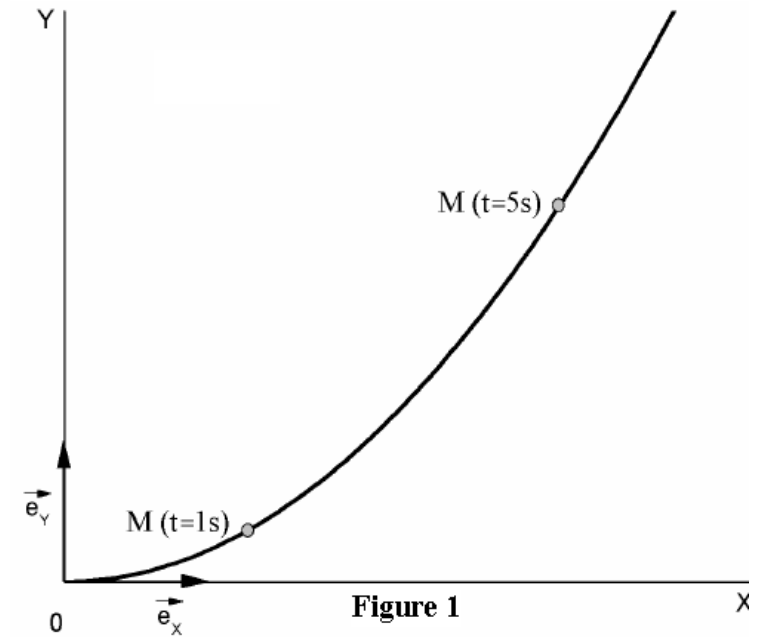
The object's equations of motion are given by the following

relations: $x = at$; $y = \frac{b}{2} t^2$; ($a = 2 \text{ m} \cdot \text{s}^{-1}$ and $b = 5 \text{ m} \cdot \text{s}^{-2}$ are constants)

where x and y represent the coordinates in the Cartesian frame $\{O, \vec{e}_x, \vec{e}_y\}$.

In this problem, the reference frame of study is attached to the axes $[Ox)$ and $[Oy)$.

- 1) Give the expression of the position vector $\overrightarrow{OM}$ in the Cartesian frame $\{O, \vec{e}_x, \vec{e}_y\}$ as a function of a , b , and *time*.
- 2) Give the equation of the trajectory of the material point as a function of a and b .⁴³



- 3) Give the expression of the velocity vector $\vec{V}$ in the Cartesian frame $\{O, \vec{e}_x, \vec{e}_y\}$ as a function of a , b , and *time*.
- 4) Give the expression of the magnitude of the velocity vector $\|\vec{V}\|$ as a function of a , b , and *time*.
Deduce the magnitude of the elementary displacement vector $\|d\vec{l}\| = ds$ as a function of a , b , t and dt .
- 5) Give the expression of the acceleration vector $\vec{a}$ as a function of a , b , and *time*.
- 6) On the figure, draw the vectors $\vec{V}$ and $\vec{a}$ at the indicated points, using the following scales:
 $1cm \rightarrow 5ms^{-1}$ for velocity, $1cm \rightarrow 2.5 ms^{-2}$ for acceleration.
- 7) Give the expressions of the Frenet basis vectors $\vec{e}_T$ and $\vec{e}_N$ as functions of a , b , t expressed in terms of the vectors $\vec{e}_x$ and $\vec{e}_y$.
- 8) Give the expression of the radius of curvature of the trajectory as a function of a , b , and *time*.

Summary

The following table presents a summary of the expressions we introduced previously.

Vector Reference(Base)	Position $\overrightarrow{OM}$	Elementary displacement $d\vec{l}$	Velocity $\vec{v}$	Acceleration $\vec{a}$
Cartesian $(\vec{i}, \vec{j}, \vec{k})$	$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$	$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$	$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$	$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}$
Polar $(\vec{e}_\rho, \vec{e}_\theta)$	$\begin{pmatrix} \rho \\ 0 \end{pmatrix}$	$\begin{pmatrix} d\rho \\ \rho d\theta \end{pmatrix}$	$\begin{pmatrix} \dot{\rho} \\ \rho \dot{\theta} \end{pmatrix}$	$\begin{pmatrix} \rho'' - \rho \theta'^2 \\ 2\rho' \theta' + \rho \theta'' \end{pmatrix}$
Cylindrical $(\vec{e}_\rho, \vec{e}_\theta, \vec{e}_z = \vec{k})$	$\begin{pmatrix} \rho \\ 0 \\ z \end{pmatrix}$	$\begin{pmatrix} d\rho \\ \rho d\theta \\ dz \end{pmatrix}$	$\begin{pmatrix} \dot{\rho} \\ \rho \dot{\theta} \\ \dot{z} \end{pmatrix}$	$\begin{pmatrix} \rho'' - \rho \theta'^2 \\ 2\rho' \theta' + \rho \theta'' \\ z'' \end{pmatrix}$
Spherical $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\phi)$	$\begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} dr \\ r d\theta \\ r d\phi \sin \theta \end{pmatrix}$	$\begin{pmatrix} r' \\ r \theta' \\ r \phi' \sin \theta \end{pmatrix}$	$\begin{pmatrix} r'' - r \theta'^2 - r \phi'^2 \sin^2 \theta \\ 2 r' \theta' + r \theta'' - r \phi'^2 \sin \theta \cos \theta \\ 2 r \phi' \theta' \cos \theta + 2 r' \phi' \sin \theta + r \phi'' \sin \theta \end{pmatrix}$
Frenet $(\vec{e}_T, \vec{e}_N, \vec{e}_B)$	$s = \widehat{OM}$	$\begin{pmatrix} ds \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} s' \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} s'' \\ \frac{v^2}{R} \\ 0 \end{pmatrix}$

7. Examples of Simple Movements

7.1. Motion in Straight Line

This is certainly the simplest type of motion of all. To analyze this type of motion, we choose the **Cartesian coordinate** system.

➤ **Uniform Rectilinear Motion (URM)**:

URM is characterized by the following conditions:

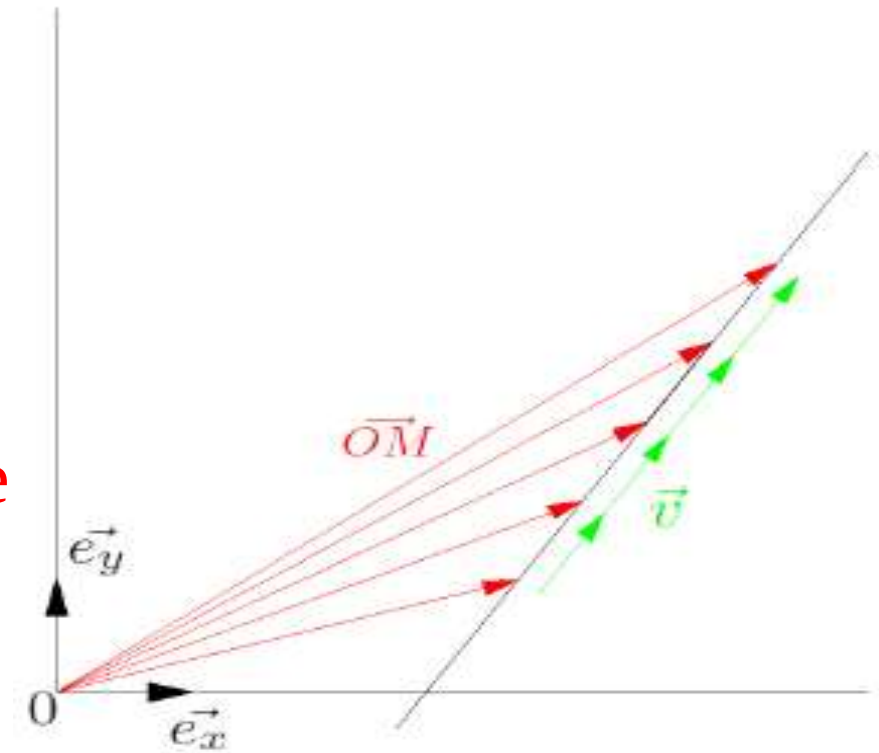
➤ The velocity is constant : $V(t) = V_0$

(neither the direction, nor the sense, nor the magnitude of the velocity vector varies over time)

➤ Therefore : $a = \frac{dV_0}{dt}$ by deriving $a = 0$

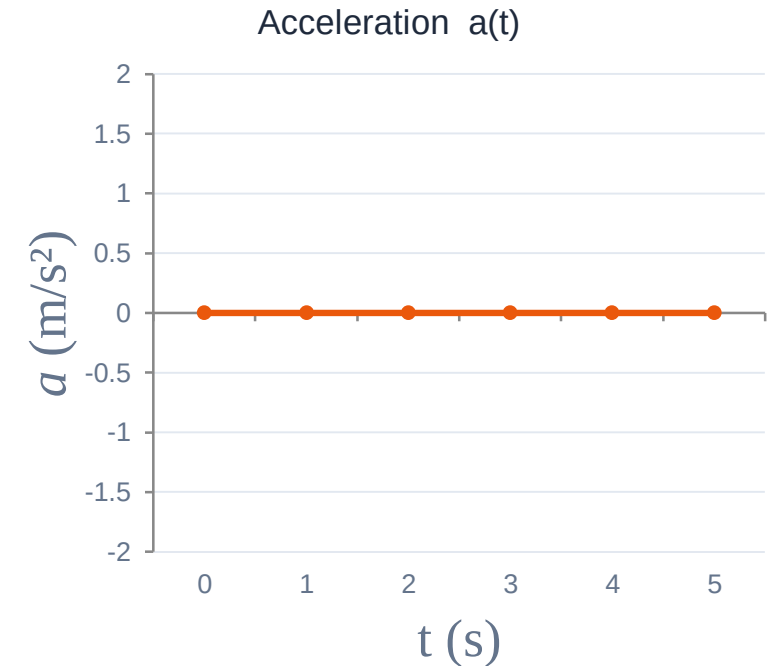
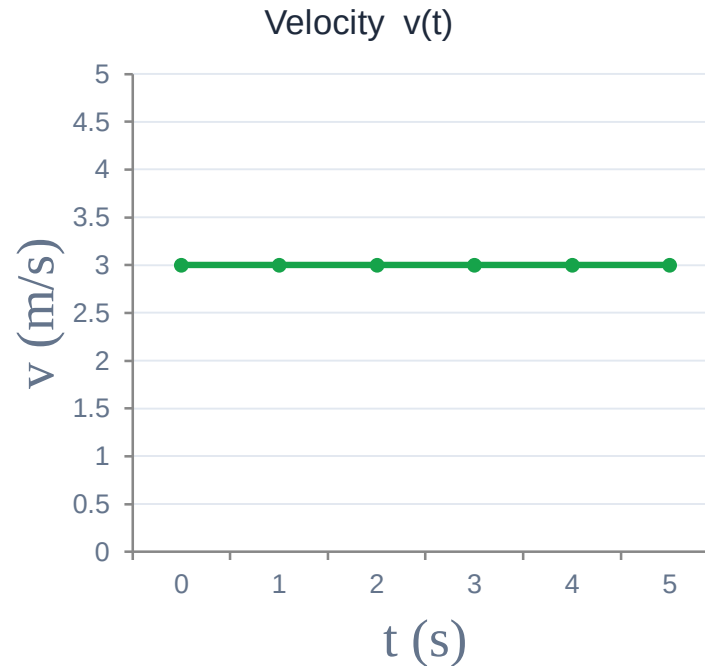
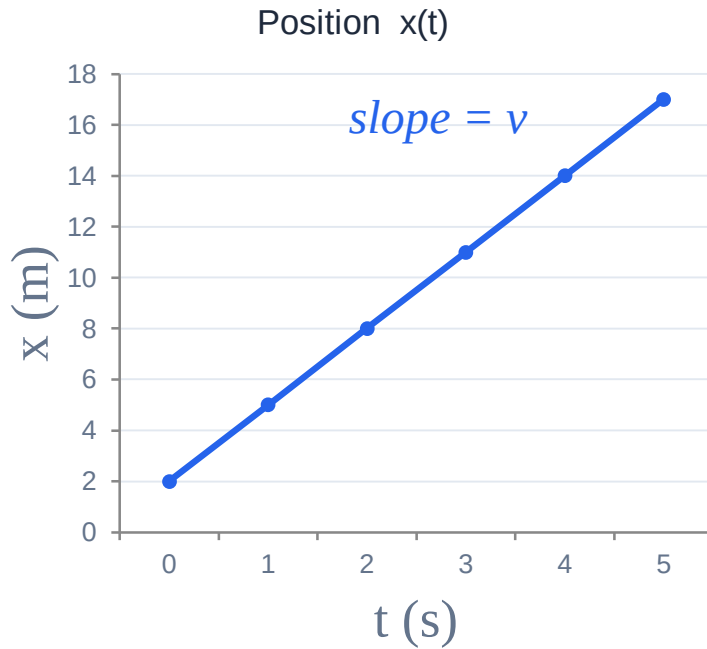
➤ $\frac{dx}{dt} = V_0$ by integrating $x(t) = x_0 + V_0 \cdot (t - t_0)$, for the **URM**

It's an equation, represented by a straight line (see figure below)



Uniform Rectilinear Motion (URM)

$$x(t) = x_0 + v \cdot t \quad \text{— constant velocity, zero acceleration —}$$



$x(t)$

Oblique straight line — slope equals velocity v

$v(t)$

Horizontal line at v_0

$a(t)$

Coincides with the time axis — acceleration is always **zero**

Example 12

The equations of motion (kinematic equations) of a material point are:

$$x = 2t$$

$$y = 2t + 4$$

$$z = 0$$

(all units are in the International System).

Show that the motion is rectilinear and uniform.

Uniformly Accelerated (Decelerated) Rectilinear Motion (URAM):

URAM (decelerated) is characterized by the following conditions:

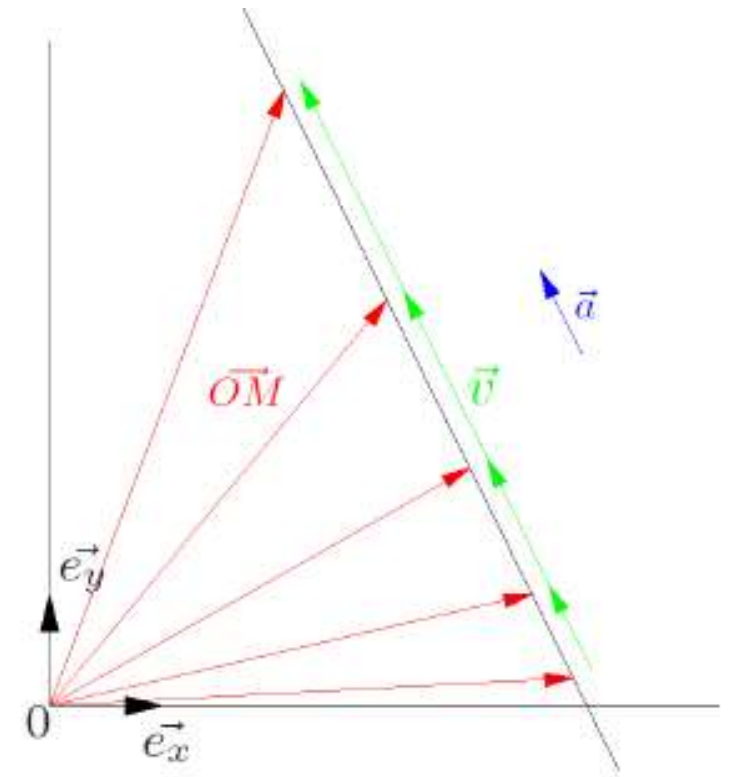
$$\blacktriangleright \vec{a} = \overline{(\text{Canstant})} = \vec{a}_0$$

$$\blacktriangleright \frac{dV}{dt} = a_0 \xrightarrow{\text{by integrating}} V(t) = V(t_0) + \int_{t_0}^t a_0 dt \Leftrightarrow \boxed{V(t) = V_0 + a_0 \cdot (t - t_0)}, \text{ for URAM}$$

$$\blacktriangleright \frac{dx}{dt} = V_0 + a_0(t - t_0) \Leftrightarrow x(t) = x_0 + \int_{t_0}^t [V_0 + a_0(t - t_0)] dt$$

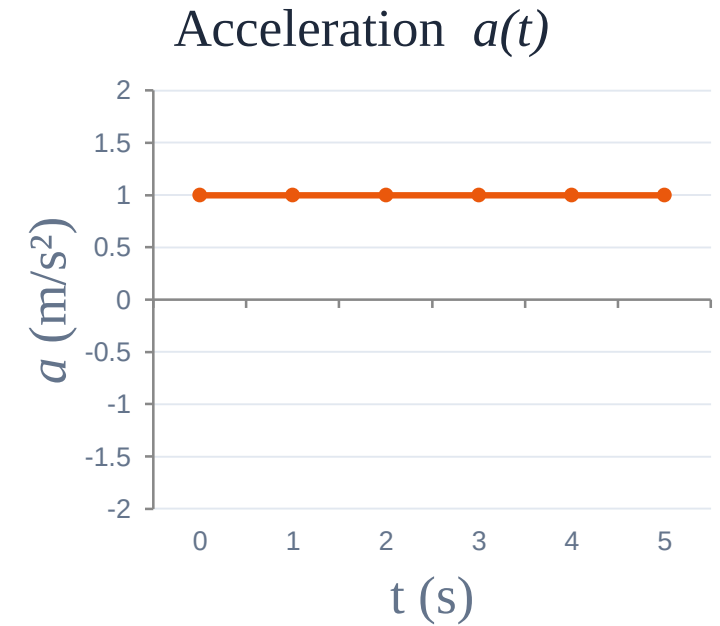
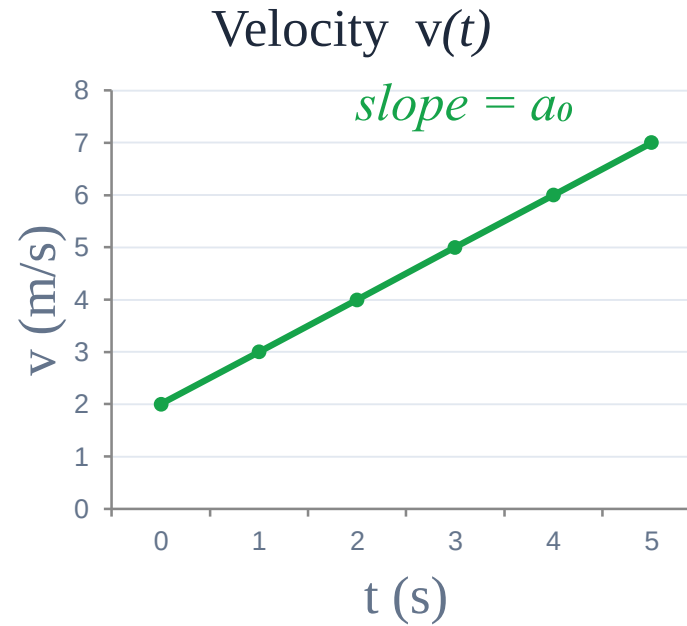
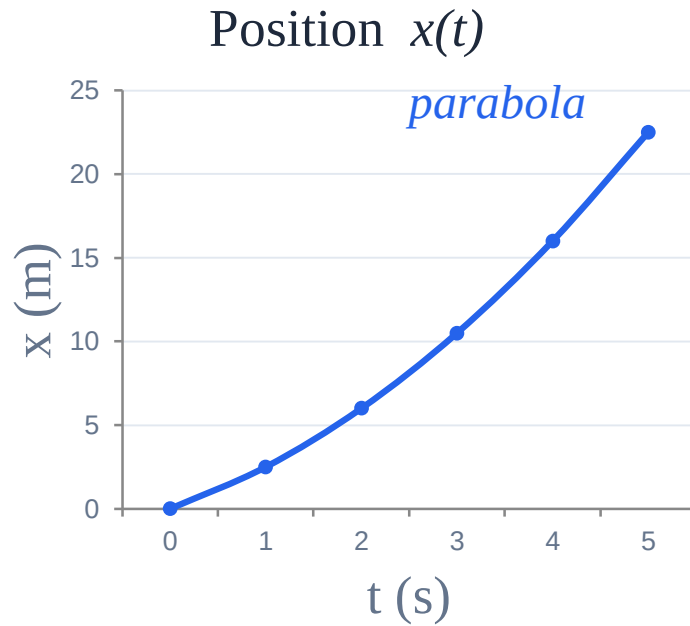
$$\boxed{x(t) = x_0 + V_0 \cdot (t - t_0) + \frac{1}{2} a_0 \cdot (t - t_0)^2}, \text{ for URAM}$$

The function $x(t)$ is of the second degree, and the curve corresponding to it is a parabola (see figure below).



Uniformly Accelerated (Decelerated) Rectilinear Motion (URAM)

$a(t) = a_0$ (*constant*) — accelerated if a_0 and V_0 have the same sign, decelerated if opposite



$x(t)$

Parabola

$$x(t) = x_0 + V_0(t - t_0) + \frac{1}{2}a_0(t - t_0)^2$$

$v(t)$

Oblique straight line — slope equals a_0

$$V(t) = V_0 + a_0(t - t_0)$$

$a(t)$

Horizontal line at a_0

By eliminating $(t - t_0)$ between the relations,

$$\begin{cases} V(t) = V_0 + a_0(t - t_0) \Leftrightarrow (t - t_0) = \frac{V(t) - V_0}{a_0} \\ x(t) = x_0 + V_0(t - t_0) + \frac{1}{2} a_0(t - t_0)^2 \end{cases}$$

we find the relation between the change in velocity and the displacement, valid only for **URAM**:

$$\begin{aligned} x(t) - x_0 &= V_0 \left(\frac{V(t) - V_0}{a_0} \right) + \frac{1}{2} a_0 \frac{(V(t) - V_0)^2}{a_0^2} \\ x - x_0 &= \frac{1}{2a_0} (V^2 - V_0^2) \\ \boxed{V^2 = V_0^2 + 2a_0(x - x_0)}, &\text{ for URAM} \end{aligned}$$

Note: Studying the sign of the product of velocity and acceleration allows us to determine whether the motion is accelerated ($\vec{V} \cdot \vec{a} > 0$) or decelerated ($\vec{V} \cdot \vec{a} < 0$).

Example 13

A point body moves along the OX axis with a velocity given by:

$$V = 2t - 6 \text{ (m s}^{-1}\text{)}; t \geq 0$$

- a) Deduce the equation of acceleration as well as the equation of motion (position-time equation), knowing that at time $t = 0, x = 5\text{m}$. What is the nature of the motion?
- b) Indicate the **accelerated** and **decelerated** phases of the **motion**.

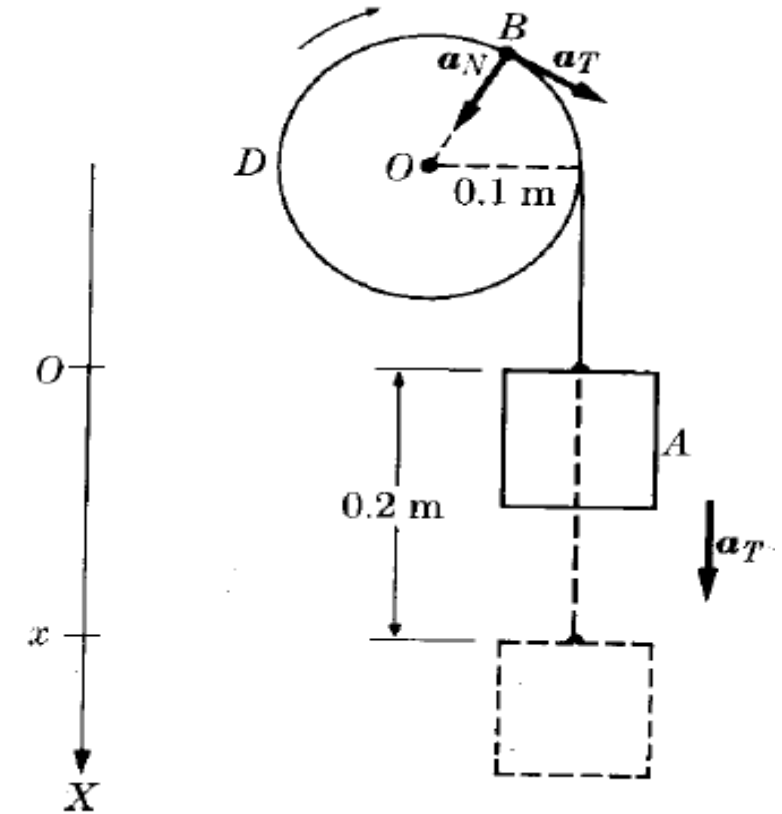
Example 14

A disk **D** with a radius of **0.1 m** is rotating freely about its horizontal axis.

A cord is wrapped around the outer circumference of the disk, and a body **A**, attached to the cord, falls under the action of gravity.

The motion of **A** is **uniformly accelerated** but its acceleration is less than that due to gravity. At $t = 0$ the velocity of body **A** is 0.04 m s^{-1} and 2 s later **A** has fallen **0.2 m**.

Find the **tangential** and **normal accelerations**, at any instant, of any point on the rim of the disk.

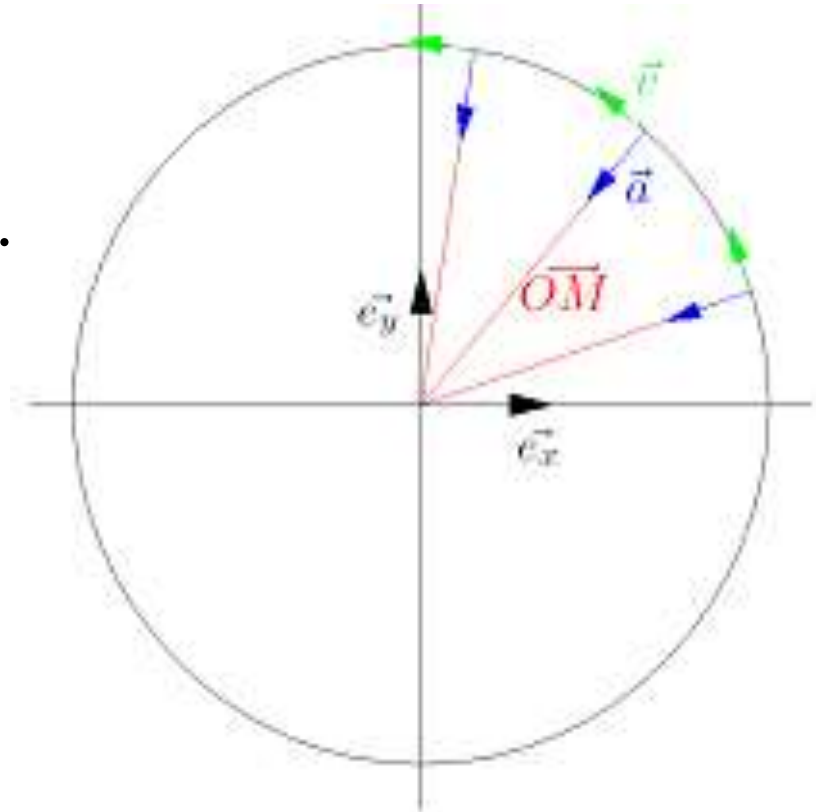


7.2. Uniform circular motion

- An object that **moves** in a **circle** with **constant speed** is in **uniform circular motion**.
- Although the **magnitude** of the **velocity** vector (**the speed**) is constant. the **direction of the velocity is changing**.
- Recall that acceleration measures the rate of change of velocity.

Uniform circular motion is characterized by the following conditions:

- $\boxed{\rho = C^{te} = \rho_0}$ (ρ_0 : radius of curvature). The magnitude of the position vector is constant. As a result, $\boxed{\frac{d\rho}{dt} = \dot{\rho} = 0}$.
- The "**angular velocity**" $\vec{\omega}$ (omega) is constant (the motion is planar); its magnitude is constant, with $\boxed{\frac{d\theta}{dt} = \dot{\theta} = C^{te} = \omega}$. As a result, $\boxed{\frac{d^2\theta}{dt^2} = \ddot{\theta} = 0}$



➤ The general expression of the velocity vector in the **polar frame** simplifies and becomes:

$$\boxed{\vec{V} = \rho_0 \omega(t) \vec{e}_\theta}.$$

➤ In this case of **uniform circular motion** only, the **polar frame** and the **Frenet frame** coincide:

$$\vec{e}_\rho = -\vec{e}_N \text{ et } \vec{e}_\theta = \vec{e}_T$$

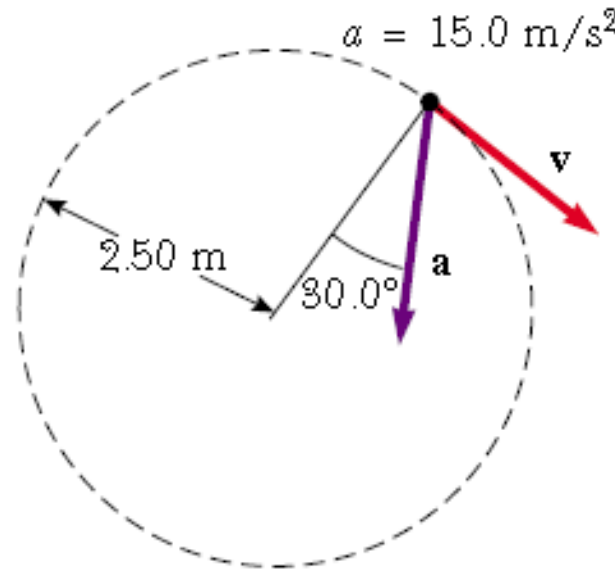
Example 15

A military jet fighter plane flying at 180 m/s pulls out of a vertical dive by turning upward along a circular path of radius 860 m .

What is the acceleration of the plane? Express the acceleration as a multiple of g .

Example 16

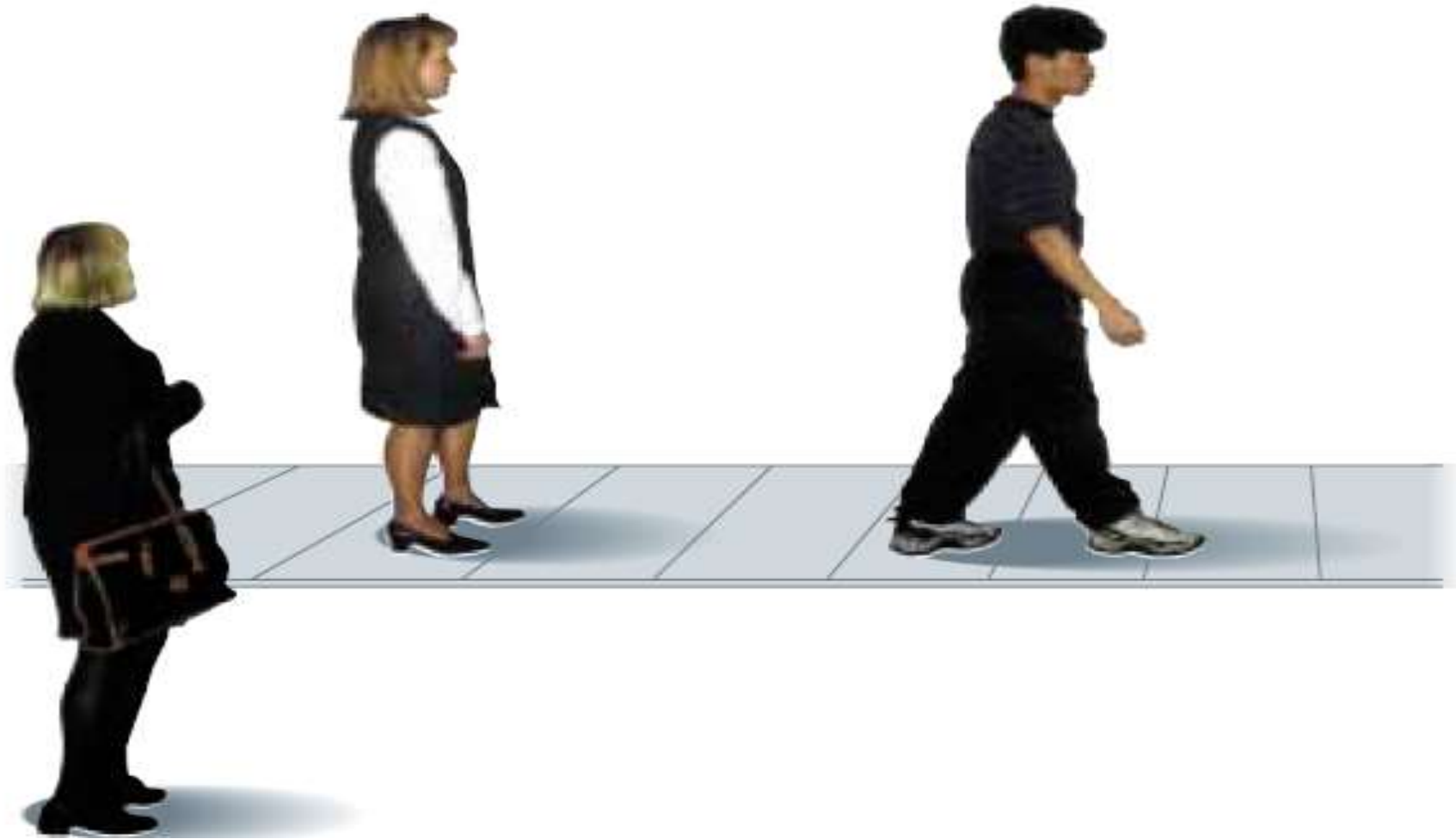
The following figure represents the total acceleration of a particle moving clockwise in a circle of radius 2.50 m at a certain instant of time.



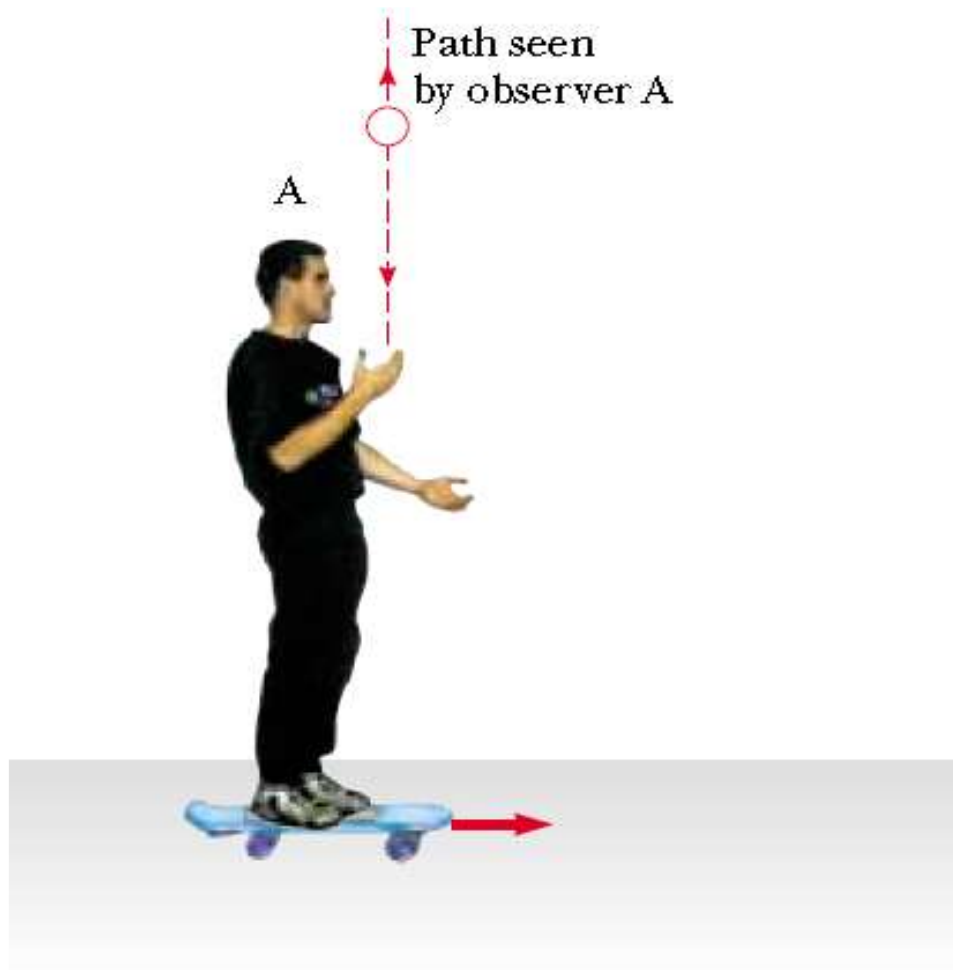
At this instant, find (a) the normal acceleration, (b) the speed of the particle, and (c) its tangential acceleration.

8. Relative Velocity and Relative Acceleration

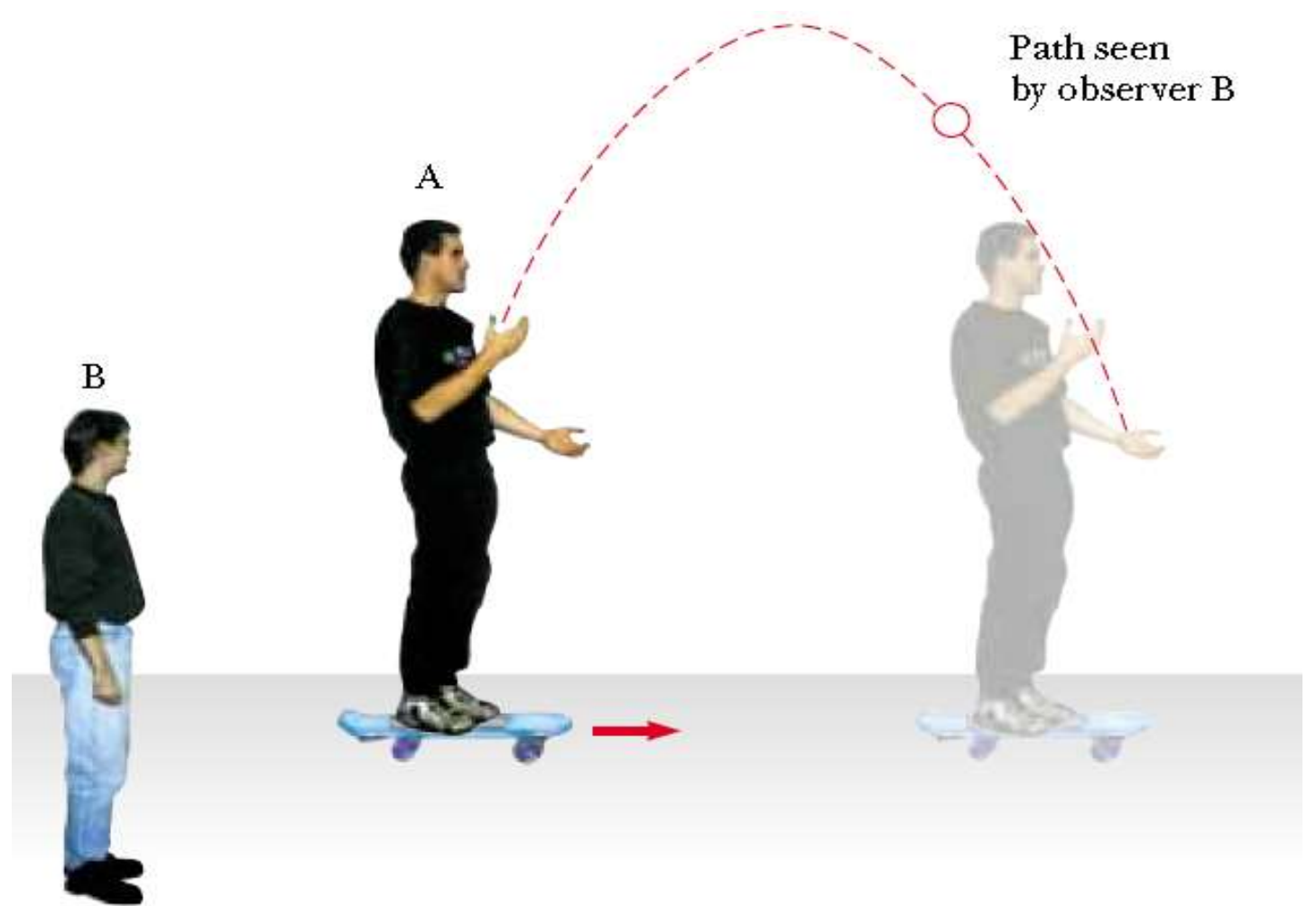
- In this section, we describe how observations made by **different observers** in **different frames** of reference are related to each other.
- We find that **observers** in different frames of reference may **measure different** positions, velocities, and accelerations for a given particle.
- That is, two observers **moving relative to each other** generally do not agree on the **outcome of a measurement**.



Two observers measure the speed of a man walking on a moving beltway. The woman standing on the beltway sees the man moving with a slower speed than the woman observing from the stationary floor.



(a)



(b)

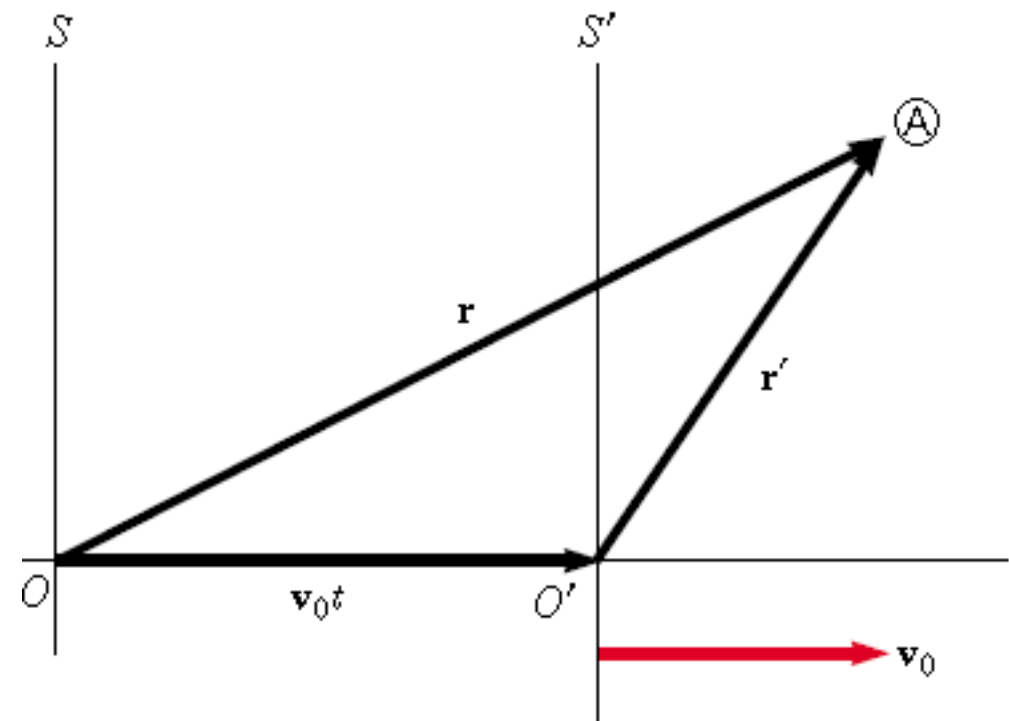
(a) Observer A on a moving skateboard throws a ball upward and sees it rise and fall in a straight-line path.

(b) Stationary observer B sees a parabolic path for the same ball.

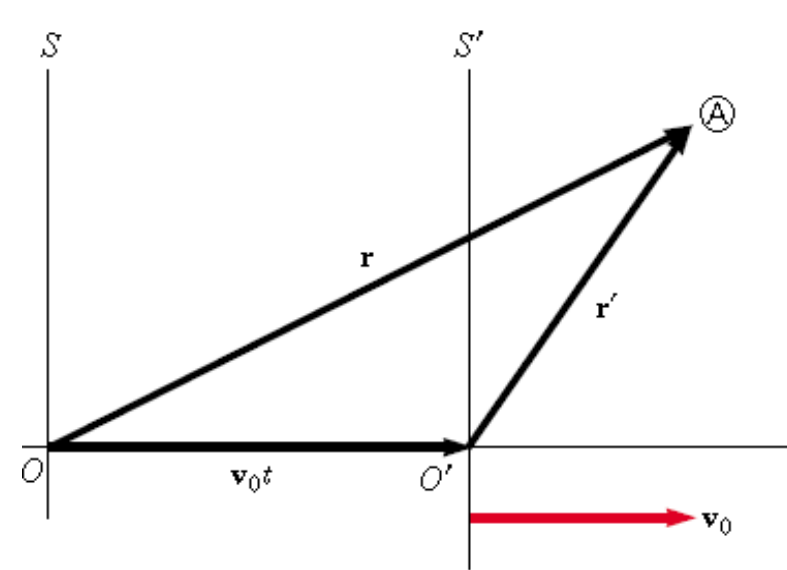
➤ Consider a particle located at point A in Figure opposite.

➤ Imagine that the motion of this particle is being described by two observers, one in reference frame S , fixed relative to Earth,

➤ and another in reference frame S' , moving to the right relative to S (and therefore relative to Earth) with a constant velocity V_0 . (Relative to an observer in S' , S moves to the left with a velocity $-V_0$)



- We define the time $t = 0$ as that instant at which the origins of the two reference frames coincide in space.
- Thus, at time t , the origins of the reference frames will be separated by a distance $V_0 t$.
- We label the position of the particle relative to the S frame with the position vector $\vec{r}$
- and that relative to the S' frame with the position vector $\vec{r}'$, both at time t .



- The vectors $\vec{r}$ and $\vec{r}'$ are related to each other through the expression $\vec{r} = \vec{r}' + V_0 t$, or $\boxed{\vec{r}' = \vec{r} - \vec{V}_0 t}$
- If we differentiate Equation $(\vec{r}' = \vec{r} - V_0 t)$ with respect to time and note that V_0 is constant, we obtain $\frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} - \vec{V}_0 \Leftrightarrow \boxed{\vec{V}' = \vec{V} - \vec{V}_0}$

where $\vec{V}'$ is the velocity of the particle observed in the S' frame and $\vec{V}$ is its velocity observed in the S frame.

➤ Equations $\vec{r}' = \vec{r} - \vec{V}_0 t$ and $\vec{V}' = \vec{V} - \vec{V}_0$ are known as Galilean transformation equations.

➤ They relate the position and velocity of a particle as measured by observers in relative motion.

➤ Although observers in two frames measure different velocities for the particle, they measure the same acceleration when $\vec{V}_0$ is constant. We can verify this by taking the time derivative of Equation $\vec{V}' = \vec{V} - \vec{V}_0$:

$$\frac{d\vec{V}'}{dt} = \frac{d\vec{V}}{dt} - \frac{d\vec{V}_0}{dt}$$

➤ Because $\vec{V}_0$ is constant, $\frac{d\vec{V}_0}{dt} = 0$. Therefore, we conclude that $\vec{a}' = \vec{a}$ because

$$\vec{a}' = \frac{d\vec{V}'}{dt} \text{ and } \vec{a} = \frac{d\vec{V}}{dt}.$$

➤ That is, the **acceleration** of the particle measured by an observer in one frame of reference is **the same** as that measured by any other observer moving with constant velocity relative to the first frame.

Example 17

A boat heading due north crosses a wide river with a speed of 10.0 km/h relative to the water. The water in the river has a uniform speed of 5.00 km/h due east relative to the Earth.

Determine the velocity of the boat relative to an observer standing on either bank.